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A Precise Unified Mass Scaling Law for Leptons and Quarks: Implications for Their Internal Geometric Structures governed by Hypercomplex Algebra

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Abstract
We propose a geometric mass scaling law for charged leptons and quarks based on a unified logarithmic formula, log m = Alog x + B + Cx that accurately describes the masses of charged leptons (x=1, 2, and 3) and light and heavy quarks (x=1^3, 2^3, and 3^3) across all generations. This simple three-parameter structure precisely reproduces experimental mass values, suggesting a deep correlation between their mass hierarchies and internal geometrical spinor structures governed by hypercomplex algebra. The distinct scaling behaviors among leptons and quarks emerge naturally from internal symmetry differences embedded in the spinor structure of 16-dimensional sedenion algebra. In contrast to a purely empirical model, our approach derives these patterns from algebraic and topological principles. The result offers insight into the three-generation structure, SU(2) versus SU(3) couplings, and mass regularities without relying on the Higgs mechanism or arbitrary parameterization.

Keywords: Mass Law, Leptons, Quarks, Sedenion Algebra, Hypersphere, Standard Model

Introduction
The Standard Model has served as the cornerstone for describing elementary particles for over half a century [1,2]. Despite its remarkable success, it leaves several fundamental unsolved problems, including the hierarchy problem the mass gap the origin of flavors and generations the origin of the Koide mass law neutrino mass the fine-structure constant parameter fine-tuning gravity and grand unification [3-14]. There is growing research interest in using various mathematical tools to develop better alternatives to the conventional Standard Model, such as geometric algebra hypercomplex algebra string theory supersymmetry grand unification theories etc [15-24,12]. In this work, we focus on addressing the mass spectrum of the three generations of leptons, light and heavy quarks. While the Standard Model relies on the experimental determination of particle masses and lacks a precise mass formula for individual particles, indirect relationships, such as those described by the Koide mass ratio formula provide some insight. However, these approaches often involve mixed-generation mass terms and do not prescribe exact mass formulas for specific particles [7,25].

Here, we propose a simple yet precise unified mass formula for the three generations of leptons and the light and heavy quark sectors. We further analyze the physical implications of this mass formula and its fitted parameters, exploring their intricate connections to hypersphere geometry characterized by the spinor structures of sedenion algebra [26]. The 16D sedenions, like 8D octonions and 4D quaternions, belong to a hyper-complex number system as a generation from the more well-known 2D complex number system to a higher dimensional space. Our recent reveals that the 16-dimensional sedenion algebra is intrinsically linked to the SU(5) symmetry group, which encompasses the Standard

Model's U(1) × SU(2) × SU(3) gauge symmetries study [22]. This sedenion-based framework extends the Standard Model by assuming that fermions are point-like Dirac particles with no internal structure or size. By providing a deeper understanding of the mass distribution of these charged particles and their potential connections to sedenion algebra, our findings offer fresh insights into the internal dynamics and symmetries of fundamental particles.

Model

This work aims to provide a precise and unified description of the masses of charged leptons, light quarks, and heavy quarks. This approach seeks to uncover the currently unknown relationships among these particles and to gain deeper insight into their internal topological structures and symmetries. Before introducing our proposed unified mass formula, we present the masses of these particles across the three generations of leptons and quarks in Table 1. Since the masses of neutrinos - the neutral counterparts of charged leptons - remain unknown, they are excluded from our analysis [27].

Classification	Particle	Experimental Mass
1st generation - Lepton	Electron (e)	0.51099895000 (15) MeV
2nd generation	Muon (μ)	105.6583755 (23) MeV
3rd generation	Tau (τ)	1776.86 (12) MeV
1st generation - Light Quark	Up (u)	2.16 (0.19) MeV
2nd generation	Down (d)	4.67 (0.48) MeV
3rd generation	Strange (s)	93.40 (0.86) MeV
1st generation - Heavy Quark	Charm (c)	1270 (20) MeV
2nd generation	Bottom (b)	4180 (0.03) MeV
3rd generation	Top (t)	17269 (500) MeV

Table 1: Masses of Charged L and Quarks

To unfold the intricate relationship among all nine particles in the above table, we seek to explore possible connections within the same family. For the charged leptons, we have found simple linear mass distribution if the lepton masses are plotted using a log-log scale with the generation index k assigned to an x-axis coordinate 1, 2, and 3, respectively. For light and heavy quarks, however, we obtain a linear appearance only if these data are plotted using a semi-log scale and the x coordinate must be assigned to a cubic k-relationship, i.e., $x = k^3$, where k is the generation index. The fitted curves using a two-parameter formula are shown in Fig. 1.

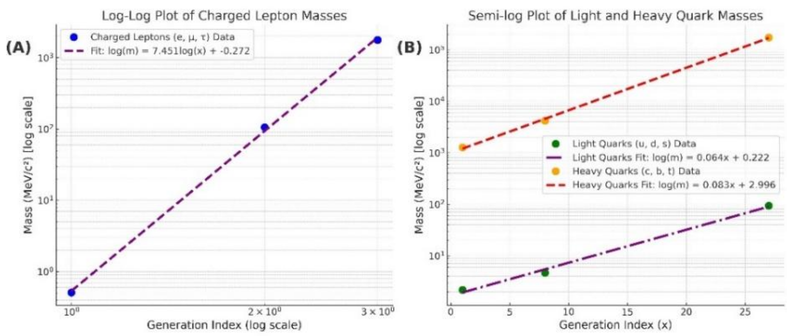

Figure 1: Linear Fits to the Masses of the Lepton, Light, and Heavy Quark Families.

To achieve a linear appearance for the data curve, in the lepton case, a log-log scale is used for both x and y coordinates, and x= 1, 2, and 3 for electron, muon, and tau, respectively. For light and heavy quarks, a semi-log scale with $x = 1^3$, 2^3, and 3^3 is required for the data curves to appear linear.

As shown in Table 1, we have found simple empirical mass-ratio formulae describing these generations of elementary particles for each sector. As illustrated in Fig. 1, we obtain a linear relation using a log-log plot with a generation index x = 1,2 and 3 for the lepton sector. However, during our initial attempt to analyze the masses of quarks, we could not obtain a simple linear line in either a log-log or semi-log plot. To achieve the goal, we must assign $x = 1^3$, 2^3, and 3^3, and use a semi-log plot. We obtain the following empirical mass formulae for the lepton, light, and heavy quark sectors. The log-log formula for leptons, the semi-log formula for quarks, and two fitting parameters with errors are shown in Table 2.

Particle Type	Fitting Formula	*A*	Errors	*B*	Errors
Leptons x = 1, 2, 3	$\log m = A \log(x) + B$	7.451	(0.192)	−0.272	(0.063)
Light Quarks $x = 1, 2^3, 3^3$	$LOG\ m = A\,x + B$	0.0641	(0.0044)	0.2233	0.0037
Heavy Quarks $x = 1, 2^3, 3^3$	$LOG\ m = A\,x + B$	0.0826	(0.0024)	2.996	0.0021

Table 2: The Two-Parameter Mass Formula for the Individual Lepton, Light, and Heavy Quark

Although the fitted lines do not pass each data point exactly, the simple linear relations capture the key characteristics and show some intricate links among three generations of particles within the same family and distinctions between lepton and quark sectors. The mass data of leptons appear linear in a log-log plot with equal spacing in x. In contrast, that data would appear linear only in a semi-log plot with a cubic x-dependence for both light and heavy quark sectors. To achieve fitting accuracy and ensure the fitted curve passes through each data point, we include a small term, and the resultant fitted curves of the mass for each sector are shown in Fig. 2.

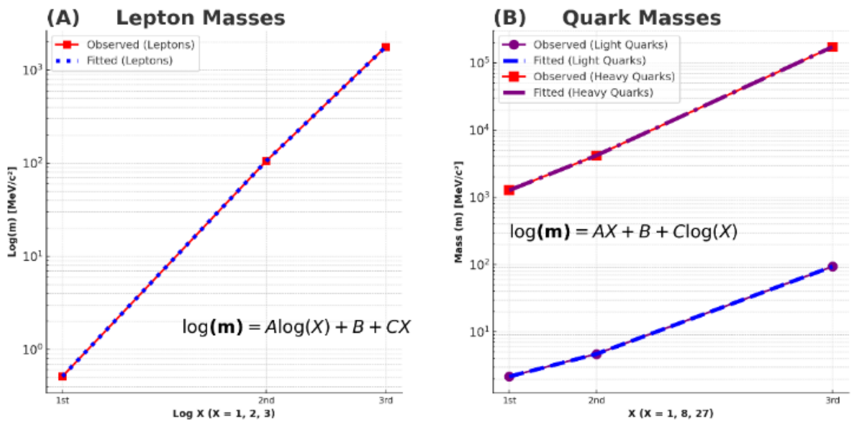

Figure 2: Three-Parameter Fits for the Masses of Lepton, Light Quark, and Heavy Quark Families

(A) The raw data and fitted curves for the lepton masses are shown on a log-log scale.
(B) The raw data and fitted curves for the masses of light and heavy quarks are displayed on a semi-log plot.

As shown in Figure 2, we employed an enhanced three-parameter mass formula to achieve greater accuracy in the fits. The fitted curves closely align with each data point, demonstrating high precision. The corresponding fitted parameters are presented in a table. Table 3 lists the improved fitting formulas and the specific parameters for each lepton, light quark, and heavy quark family. The mass formulas for the lepton and quark sectors are fundamentally similar, with a key distinction: the parameters A and C are interchanged, reflecting the primary role of the A term. In the discussion section, we will delve into the physical implications of this dependency and explore the underlying reasons for the intriguing cubic x-dependence on the generation index.

Category	Fitting Formula	A	B	C
Lepton (e, μ, τ) x = 1, 2, 3	$\log m = A \log x + B + Cx$	8.7222	0.01856	-0.3101
Light Quark (u, d, s) $x = 1, 2^3, 3^3$	$\log m = A\,x + B + C \log x$	0.0741	0.2603	-0.2039
Heavy Quark (c, b, t) $x = 1, 2^3, 3^3$	$\log m = A\,x + B + C \log x$	0.0881	3.0157	-0.1101

Table 3: The Three-Parameter Mass Formula and Fitted Parameters

Using the above Table, we can determine the mass ratio between any pair of particles belonging to the same family or cross the family from the above mass formula and reactions. First of all, we could determine the average slope S, or define as the scaling factor for leptons as $S_{lepton} = \frac{d\, log\, m}{d\log x}$=A+2C ln 10 = $7.294 \sim \sqrt{5}\ \pi^2/3 \sim = 7.356.$

For the quarks, due to the cubic x-dependence, one needs to consider this complication to calculate the effective scaling factor. With $y = Ax + B + C\log x = A\xi^3 + B + 3C\log\xi,$ one has $S = \frac{d\log m}{d\log\xi} = 3A\ln 10 < \xi^3 >_{ave} + 3C$ = A(1+8+27) ln 10 +3C , S_{light} = 5.531 ~ $\sqrt{5}\ \pi^2/4 = 5.517.$ For the heavy quarks S_{heavy} = 6.973 ~ $\pi^2/\sqrt{2}$ = 6.979. One has $S_{heavy}/\ S_{lepton} = 3/4$ and $S_{light}/\ S_{heavy}$ = $0.793 \sim \sqrt{10}\ /4$ = 0.791. The simple relations to π² of the scaling factors for each family of the leptons and quarks seem to imply some deeper connections to the 4D hypersphere [28] with a volume V_4 = π²/ 2. The above results of the scaling factor S are summarized in Table 4.

Category	Fitting Formula	A	B	C	Scaling factor S
Lepton (e, μ, τ) x = 1, 2, 3	$m = x^A\ 10^{B+Cx}$ $m_e = 10^{B+C}$ $m/m_e = x^A\ 10^{C(x-1)}$	8.7222	0.01856	-0.3101	7.294 $\sim \sqrt{5}\pi^2/3$
Light Quark (u, d, s) x = κ³ κ = 1, 2, 3	$m = e^{Axln10}10^B\ x^C$ $m = e^{Axln10}10^B\ x^C$ $= \exp(A\ k^3 ln10)\ 10^B\ k^{3C}$ $m_u = exxp^{(A\ ln10)}\ 10^B$ $m/m_u =$ $\exp(A\ (k^3-1)1ln10)\ k^{3C}$	0.0741	0.2603	-0.2039	5.531 $\sim\sqrt{5}\ \pi^2/4$
Heavy Quark (c, b, t) x = κ³ κ = 1, 2, 3	$m = e^{Axln10}10^B\ x^C$ $= \exp(A\ k^3 ln10)\ 10^B\ k^{3C}$ $m_C = exp(A\ ln10)\ 10^B$ $m/m_c =$ $\exp(A\ (k^3-1)1ln10)\ k^{3C}$	0.0881	3.0157	-0.1101	6.973 $\sim \pi^2/\sqrt{5}$

Table 4: The Mass and Mass-Ratio Formulae, and Scaling Factor

From the mass formula and fitted parameters given in Tables 3 and 4, one obtains the electron's mass as $m = 10^{B+C} = 10^{-0.29154}$ = 0.51105, the up quark's mass as $m = 10^{A+B} = 10^{A+B} = 10^{0.3344}$ = 2.160, and the up quark's mass as $m = 10^{A+B} = 10^{3.1038}$ = 1270.06, respectively. The scaling factor S for each lepton, light, and heavy quark sector is shown in the above table to be related to the 4D hypersphere. The agreement with the experimental values validates again that our mass formula is precise. In the next section, we shall discuss the links of the scaling factor to 4-D hypersphere geometry and explain the cause of the interesting cubic x-dependence of the masses for quarks. '

Discussion and Conclusions

One can determine the mass ratio between any pair of particles belonging to the same family or cross the family from the above mass formula and reactions. First of all, we could determine the average slope, or define as the scaling factor for leptons as $S_{lepton} = \frac{d\, log\, m}{d\log x}$=A+2C ln 10 = $7.294 \sim \sqrt{5}\ \pi^2/3 \sim = 7.356.$ For the quarks, due to the cubic x-dependence, one needs to consider this complication to calculate the effective scaling factor. With $y = Ax + B + C\log x = A\xi^3 + B + 3C\log\xi$, onehas $S = \frac{d\log m}{d\log\xi} = 3A\ln 10 < \xi^3 >_{ave} + 3C$ = A(1+8+27) ln 10 +3C , S_{light} = 5.531 ~ $\sqrt{5}\ \pi^2/4 = 5.517.$ For the heavy quarks S_{heavy} = 6.973 ~ $\pi^2/\sqrt{2}$ = 6.979. $S_{heavy}/\ S_{lepton} = 3/4$ and $S_{light}/\ S_{heavy} = 0.793 \sim \sqrt{10}\ /4$ = 0.791. One has The simple relations to π^2 of the scaling factors for each family of the leptons and quarks seem to imply some deeper connections to the 4D hypersphere with a volume $V_4 = \frac{\pi^2}{2}$ [26] .

Some may raise concern that our model uses nine parameters for nine particles and therefore lacks predictive power. However, this is a misunderstanding of the structure. Our model applies a **single functional form with three parameters per sector,** not one parameter per particle. The same formula is reused for each family (leptons, light quarks, and heavy quarks), and its simplicity and universality strongly constrain the fits. Arbitrary parameterizations generally do not yield the observed log-linear or log-cubic behavior with such precision, nor do they reveal geometric scaling relations connected to π^2 or hypersphere volumes. Moreover, our formulation is guided by internal symmetry differences (SU(2) vs. SU(3)) and sedenion spinor structure, offering a geometric interpretation—not an arbitrary numerology. While not yet derived from first-principles dynamics, the fitted scaling patterns suggest a deeper internal organization consistent with compactified higher-dimensional spinor modes.

After the above analysis of the mass formula and the scaling factor, we now explore the links of sedenion mathematical structure to the 4D hypersphere geometry and the cubic x-dependence in the mass formula for quarks. In our recent sedenion study we showed that the sedenion algebra could be linked to SU(5), which contains $U(1) \times SU(2) \times SU(3)$ of the Standard Model [22]. Here, we make a brief introduction of the sedenion mathematical structure. Using the Cayley-Dickson construction scheme one can construct 2D complex numbers from a pair of 1D real numbers, Hamilton's 4D quaternions from a pair of complex numbers, and Cayle"s 8D octonions from a pair of quaternions, and 16D sedenions from a pair of octonions [28]. These hypercomplex algebras have found many applications in special relativity, Maxwell equation, relativistic quantum theory, and particle physics. The sedenion algebra contains sixteen basis elements, $\{e_0, \mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_{15}\}$, where e_0 is the identity basis element, and all others are anti-commutative following a set of multiplication rules. For better distinction among these elements, we define these operators as one quaternion t $\{\mathbf{I}, \ \boldsymbol{\Gamma}_1, \boldsymbol{\Gamma}_2, \boldsymbol{\Gamma}_3\}$, and three quartets with four anti-commutative basis elements, $\{\boldsymbol{\Theta}_1, \mathbf{U}_1, \mathbf{U}_2, \mathbf{U}_3\}$, $\{\boldsymbol{\Theta}_2, \mathbf{V}_1, \mathbf{V}_2, \mathbf{V}_3\}$, and $\{\boldsymbol{\Theta}_3, \mathbf{W}_1, \mathbf{W}_2, \mathbf{W}_3\}$, respectively. All sets contain a SU(2) spinor pseudo-time in these three quartets, each includes a pseudo-time anti-commutative operator, unlike the quaternion set with identity operators. These three quartets play an important role in contributing to three generations of leptons and quarks. The mathematical structure of sedenions is illustrated in Figure 3.

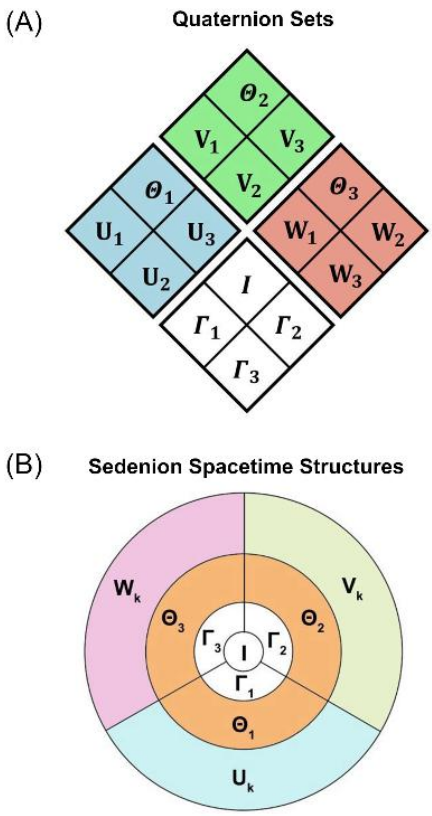

Figure 3: The mathematical structure of 16-element sedenion algebra. (A) The schematic diagram showing sedenions consist of a quaternion and three quaternion-like anti-commutative quartets, $\{\Theta_1, U_1, U_2, U_3\}$, $\{\Theta_2, V_1, V_2, V_3\}$, and $\{\Theta_3, W_1, W_2, W_3\}$, representing internal spacetime. These three quartets describe the 12D internal structure dynamics of elementary particles. (B) The schematic diagram shows three 8-element octonion subsets, each sharing a common quaternion $\{I, \Gamma_1, \Gamma_2, \Gamma_3\}$, representing the 4D exterior Minkowski spacetime.

As shown in Figure 3, the 16D sedenion algebra includes a 4D quaternion set, representing the external 4D Minkowski spacetime, and three additional 4D quartets that collectively describe the internal spacetime, amounting to twelve degrees of freedom. For a quantized n-dimensional structure composed of unit cells, the total energy correlates with the sum of the fundamental mode energies of each basic unit cell. According to Einstein's special relativity and relativistic quantum theory, 3D space and 1D time are treated on an equal footing, forming a unified 4D spacetime, with each axis contributing one degree of freedom.

For a quantized 4D unit cell, it becomes evident why the scaling factor SSS for the mass increment rate, as shown in Table 4, is linked to 4D hypersphere geometry. This geometry reflects the interplay between the 4D quaternion Minkowski spacetime and the three 4D quartets representing the internal structure. The distinct cubic dependence of the x^3-coordinate on the generation index for quarks versus the linear x-dependence for leptons arises from their respective internal symmetries: the SU(3) symmetry for quarks and the SU(2) symmetry for leptons.

In our previous report, we highlighted that the sedenions encompass three distinct octonion subsets, each sharing a common quaternion set $\{I, \Gamma_1, \Gamma_2, \Gamma_3\}$. From the first octonion subset, the generalized Dirac equation for the electron, formulated using octonions, the internal space consists of an SU(2) symmetry group $\{U_1, U_2, U_3\}$ which is coupled to external spacetime via the photon—a vector boson with an electromagnetic (EM) field represented by $b\{\Gamma_1, \Gamma_2, \Gamma_3\}$, where b denotes the creation or annihilation operator of a scalar boson. Similarly, for the muon, representing the second generation of leptons, an additional spinor set $\{V_1, V_2, V_3\}$ is included for coupling to the photon EM field. For the tau, the third generation, yet another spinor set $\{W_1, W_2, W_3\}$ comes into play. This implies the coupling for the electron to the EM field involves the Uk triplet, and the muon with a coupling with both U_k and V_k triplets, and the tau with U_k, V_k, and W_k triplets. Consequently, as the generation index increases, the lepton mass formula exhibits a linear dependence on x, where x = 1, 2, 3 corresponds to the first, second, and third lepton generations, respectively.

In that work, we also demonstrated that three pairs of fermion creation and annihilation operators can be constructed: one pair, $c_3^+ = \frac{U_3+i\Gamma_2}{2}, c_3 = \frac{-U_3+i\Gamma_2}{2}$, derived from inter-mixing between the spinor sets $\{\Gamma_1, \Gamma_2, \Gamma_3\}$ and $\{U_1, U_2, U_3\}$, and one pair, $c_1^+ = \frac{U_2+iU_1}{2}, c_1 = \frac{-U_2+iU_1}{2}$, from the intra-mixed U_k triplet, and the other pair, $c_2^+ = \frac{\Gamma_3+i\Gamma_1}{2}, c_2 = \frac{-\Gamma_3+i\Gamma_1}{2}$, with the intra-mixed Γ_k triplet. Using these three pairs of fermion operators, one can construct a 3 by 3 matrix $M_{jk} = c_j^+ c_k$, and then use matrix M to construct eight SU(3) generators. These Gell-Mann's lambda matrices are given by $\Lambda_1 = M_{12}+ M_{21}$, $\Lambda_2 = -i M_{12} + iM_{21}$, $\Lambda_3 = M_{11} - M_{22}$, $\Lambda_4 = M_{13}+ M_{31}$, $\Lambda_5 = -i\ M_{13} + iM_{31}$, $\Lambda_6 = M_{23}+ M_{321}$, $\Lambda_7 = -i\ M_{23}+ iM_{32}$, and $\Lambda_8 = (M_{11} + M_{22} - 2M_{33}) / \sqrt{3}$. Because these SU(3) generators are the foundation for strongly interacting gluons and quarks, the inter-mixing between the Γ_k operators for the exterior Minkowski space with U_k of the particle's internal space explains why gluons and quarks, unlike the photon, electron or other leptons, are spatially confined internally.

To construct first-generation quarks, the spinor set $\{U_1, U_2, U_3\}$ must couple with gluons, and there is only one possible arrangement for this coupling. For the second-generation quarks, the inclusion of the spinor set $\{V_1, V_2, V_3\}$ and the gluon's mixing dynamics provides 23 possible configurations for constructing a three spatial-component vector (X, Y, Z) from combinations of $\{U_k, V_k\}$. For third-generation quarks, this increases to 33 possible configurations among $\{U_k, V_k, W_k\}$ of (X, Y, Z). A fundamental distinction between leptons and quarks lies in their construction. For photons and leptons, there is no inter-mixing between the sets $\{\Gamma_1, \Gamma_2, \Gamma_3\}$and $\{U_1, U_2, U_3\}$ allowing photons and leptons to propagate freely in Minkowski space. In contrast, the construction of gluons and quarks involves inter-mixing between these sets, leading to gluon and quark confinement within the internal space [22]. Furthermore, lepton construction utilizes complete spinor triplets $\{U_k, V_k, W_k\}$ resulting in a linear dependence on the generation index x in the lepton mass formula. By contrast, quark construction requires mixing among different spinor triplets, producing a cubic dependence on x^3 for the generation index in the quark mass formula.

Summary

In summary, we present a detailed analysis of the masses of charged leptons and quarks, offering a simple unified formula that precisely matches their experimental values. Unlike the Koide mass-ratio formula or other approaches focused on mass ratios, our method introduces a novel perspective to obtain a direct and universal formula [29-31]. This approach provides deeper insights into the physical implications of the mass spectrum, the intricate patterns associated with the generation index, and the fundamental differences between leptons and quarks. Our analysis connects the fitted parameters and mass distribution to hypersphere geometry and sedenion algebra. While we do not derive the masses of leptons and quarks from first principles, the proposed mass formula—coupled with a qualitative explanation for the linear x-dependence of leptons and the cubic x3-dependence of quarks—offers some insights. These insights could help uncover the connections between particle masses, their internal structures, and underlying symmetries.

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Author Contributions

JT initiated the project, conceived the model, derived the equations, and wrote the manuscript. QT prepared the figure and reference list.

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