ElectroSphere: Electrical Electronics Engineering Bulletin ISSN: 3068-9155

PRIME OPEN ACCESS

Volume 1, Issue 2

Research Article

Date of Submission: 25 Sep, 2025
Date of Acceptance: 14 Nov, 2025
Date of Publication: 21 Nov, 2025

About the Ultimate Efficiency of Interference Binary Codes

Adzhemov Artem Sergeevich*

MTUCI, President - Chairman of the MTUCI Board of Trustees, Doctor of Technical Sciences, Professor, Moscow, Russia

***Corresponding Author:**
Adzhemov Artem Sergeevich, MTUCI, President - Chairman of the MTUCI Board of Trustees, Doctor of Technical Sciences, Professor, Moscow, Russia.

Citation: Sergeevich, A. A. (2025). About the Ultimate Efficiency of Interference Binary Codes. *Electro Sphere Electr Electronic Eng Bull, 1*(2), 01-11.

Abstract
The article presents the results of a study on the detection of error-correcting coding methods that provide the maximum coding rate for various values of the minimum coding distance from 1 to 20.

Annotation
Modern infocommunication systems, combining the capabilities of computer technology and communication devices, almost always have a digital implementation using a binary code. At the same time, various code constructions are used, which, on the one hand, provide high efficiency of transmission of initial messages, the required noise immunity and, if possible, simple practical implementation, which makes it possible to achieve lower technical and financial costs, minimize delay, etc. In this case, the question of how much the solutions used differ from the theoretically achievable potential boundaries is very important.

Keywords: Code Construction, Minimum Code Distance, Noise Immunity, Coding Efficiency, Theoretically Achievable Boundary, Concatenated Codes

Introduction
Methods and constructions of error-correcting coding, which have very diverse implementations, are constantly being improved taking into account their correcting properties, the complexity of practical implementation, delay time, etc. [1-4]. Let us examine a number of these properties in relation to binary code.

A number of authors have obtained expressions that estimate the maximum achievable boundaries that determine the guaranteed multiplicity of detected and corrected errors, based on the possible minimum code distance d_{min}.

In the theory of error-correcting coding, a number of potential boundaries are known, among which we will point out the Hamming boundary [5]:

$$2^k \le \frac{2^n}{\sum_{i=0}^{\frac{d_{min}-1}{2}} C_n^i} \text{ or } \ge \log_2\left(\sum_{i=0}^{\frac{d_{min}-1}{2}} C_n^i\right), \quad (1)$$

The Plotkin boundary:

$$d_{min} \le \frac{n \cdot 2^{k-1}}{2^k - 1} \text{ or } r \ge 2 \cdot (d_{min} - 1) - \log_2 d_{min} \text{ for } n \le 2 \cdot d_{min} - 1\,, \quad (2)$$

Varshamov - Hilbert boundary:

$$r \geq \log_2\left(\sum_{i=0}^{d_{min}-2} C_{n-1}^{i}\right), \tag{3}$$

Singleton boundary:

$$d_{min} \leq n - k + 1\ . \tag{4}$$

In formulas (1-4), the value n is the length of the binary codeword, k is the number of information symbols in the codeword, r is the number of check symbols. Wherein:

$$n = k + r. \tag{5}$$

The indicated boundaries, for example, from formulas (1, 2, 4), denote the upper limit, above which there are no error-correcting codes with a given parameter d_{min}. The Varshamov-Hilbert estimate (3) establishes a lower bound.

Limit achievable boundaries of the efficiency of error-correcting coding

In a method was proposed for constructing an error-correcting code with limiting performance characteristics for any given value of the minimum code distance d_{min} [6]. Exact expressions are obtained for the number of information symbols k depending on the length of the codeword n at d_{min}= 2, 3, 4. Let's indicate these expressions:

$$k = n - 1 \tag{6}$$

for d_{min}=2;

$$k = n - 1 - \lfloor log_2\, n \rfloor \tag{7}$$

for d_{min}=3;

$$k = n - 2 - \lfloor log_2(n-1) \rfloor \tag{8}$$

for d_{min}=4.

If we recall that for d_{min}=1, the number k=n then we can notice that the value of k determined by some formula derived for odd values of d_{min}, for the next even value of d_{min} will be calculated using a similar formula by replacing the value of n with (n-1).

Similarly to formulas (6-8), we indicate the number of information symbols arising from the Hamming boundary:

$$k \leq n \tag{9}$$

for d_{min}=2;

$$k \leq n - \log_2(n+1) \tag{10}$$

for d_{min}=3 and 4.

Comparison shows that the Hamming bound gives slightly overestimated estimates relative to those actually achievable, especially in cases where the minimum code distance is an even number.

Let us also make a comparison with similar expressions obtained on the basis of the Singleton bound. We have:

$$k \leq n - 1 \tag{11}$$

for d_{min}=2;

$$k \leq n - 2 \tag{12}$$

for d_{min}=3;

$$k \leq n - 3 \tag{13}$$

for d_{min}=4.

The Singleton boundary for d_{min}=2 coincides with the exact estimate given in (6), while for other values of d_{min}=3,4 this estimate is overestimated, which follows from a comparison of (12) and (7), as well as (13) and (8).

To obtain estimates of efficiency at large values of d_{min}, computer calculations were carried out according to the following considerations.

For d_{min}=1, all code combinations of n symbols, the number of which is equal to $N_n = 2^n$, meet this condition. For d_{min}=2, as shown in, all code words of n symbols can be decomposed into exactly two groups that meet this condition and consisting of $K_n^{d_{min}} = K_n^2 = 2^{n-1}$ of various code combinations [7]. However, at large values of d_{min}, finding the code combinations that meet this condition becomes more and more difficult.

Let $R_n^{d_{min}}$ denote the number of groups into which code combinations of n symbols are decomposed for a given value of d_{min}. At the same time, it is obvious that:

$$N_n = K_n^{d_{min}} \cdot R_n^{d_{min}} \tag{14}$$

The code combinations included in this group will be called allowed, the rest - prohibited. At the same time, we note that forbidden combinations for one group will be allowed for another group of the same dimension and meeting the d_{min} condition within its group.

Let us agree on a group of allowed code combinations, among which there will be a zero code combination, to call the main one, write it in the form of a matrix of dimension $K_n^{d_{min}} \times n$ and denote it by A_n. All other groups will be written in the form of matrices of the same dimension, called adjacent to the main A_n and denoted as B_n^i, where $i = \overline{1,\ (d_{min} - 1)}$, indicates the minimum code distance between this adjacent matrix and main.

Obviously, in the main group, all code words, except zero, with weight $w \geq d_{min}$.

For d_{min}=1 and d_{min}=2, as n grows, the number of allowed code combinations $K_n^{d_{min}}$ in the group monotonically increases and the number of groups remains unchanged $R_n^{d_{min}} = R_n^1 = 1$and $R_n^{d_{min}} = R_n^2 = 2$. However, as d_{min} grows, the situation changes and becomes very complicated. Let's investigate this.

If the value d_{min} is specified, then for the first time there are exactly $K_n^{d_{min}} = 2$ allowed code combinations when $n = d_{min}$. It is easy to show that the number of allowed code words will remain equal to $K_n^{d_{min}} = 2$ for all values

$$n = \overline{d_{min},\ (d_{min} - 1 + \lceil d_{min}/2 \rceil)} \tag{15}$$

and only then, with a further increase in n, the increase in the values of $K_n^{d_{min}}$ will begin.

Since relation (14) must be satisfied, then under conditions when n increases, and the value $K_n^{d_{min}}$ remains unchanged, the number of groups $R_n^{d_{min}}$ will increase, each of which will have its own spectrum of weights (w) of allowed code combinations included in this group.

The quantity $K_n^{d_{min}}$ can vary as a power of two, i.e. we can write that $K_n^{d_{min}} = 2^k$. Then, as shown in among the possible code combinations of n symbols, one can single out $R_n^{d_{min}} = 2^r$ different disjoint groups of allowed code combinations consisting of 2^k combinations that satisfy the requirement $d \geq d_{min}$ [7]. Moreover, it is obvious that $n = k + r$.

It is also shown in that if for some n a basic group A_n of $K_n^{d_{min}} = 2^k$ allowed code combinations is defined, then for combinations consisting of n +1 symbols, there is a basic group A_{n+1}) of 2^{k+1} allowed code words, if among the codewords of n symbols there is a contiguous group $B_n^{d_{min}-1}$ of $K_n^{d_{min}} = 2^k$ code words with a minimum code distance d_{min} that differ from combinations of group A_n by a code distance equal to (d_{min}-1) [7]. In this case, the number of allowed code combinations will be equal to $K_{n+1}^{d_{min}} = 2^{k+1}$.

If the adjacent matrix $B_n^{d_{min}-1}$ does not exist for certain n, then one should continue the sequential search for adjacent matrices $B_n^{d_{min}-2}$, $B_n^{d_{min}-3}$ etc. After detecting the existence of the first adjacent matrix $B_n^{d_{min}-i}$, where i varies from 1 to (d_{min}-1)to construct the next main matrix A_{n+i} based on the known matrix A_n and the found $B_n^{d_{min}-i}$ recurrence formula should be used:

$$A_{n+i} = \begin{Bmatrix} 0_1 0_2 & & 0_i & A_n \\ 1_1 1_2 & \cdots & 1_i & B_n^{d_{min}-i} \end{Bmatrix}, \qquad (16)$$

where the value $i = \overline{1,\ (d_{min}-1)}$, and the number of columns-zeros (0_i) and columns-ones (1_i) of dimension $K_n^{d_{min}} \times 1$ assigned to the left is i.

In this case, the construction of A_{n+i} must occur in a sequential search for the adjacent matrix $B_n^{d_{min}-i}$ with the values $i = \overline{1,\ (d_{min}-1)}$. In other words, when the matrix A_n is known, an attempt is made to find $B_n^{d_{min}-1}$ and, if such a matrix exists, then the matrix A_{n+1}. is constructed.

If the matrix $B_n^{d_{min}-1}$ does not exist, then the same number of allowed code combinations remains in the matrix A_{n+1} i.e. $K_{n+1}^{d_{min}} = K_n^{d_{min}}$, but the number of adjacent groups doubles, i.e. $R_{n+1}^{d_{min}} = 2 \cdot R_n^{d_{min}}$, since equality (3) must be satisfied. Next, the matrix $B_n^{d_{min}-2}$ is searched for, and when it is found, the matrix A_{n+2} is constructed according to formula (16). If the matrix $B_n^{d_{min}-2}$ also does not exist, then the matrix A_{n+2} also contains $K_{n+2}^{d_{min}} = K_n^{d_{min}}$ of allowed code combinations, but the number of adjacent groups, according to (14), doubles, i.e. $R_{n+2}^{d_{min}} = 2 \cdot R_{n+1}^{d_{min}} = 4 \cdot R_n^{d_{min}}$. Etc.

It should be noted that in the situations considered, when it is not possible to find the adjacent matrix $B_n^{d_{min}-1}$ and then $B_n^{d_{min}-2}$, additional (n+1) and (n+2) digits are not used to expand the number of allowed code combinations and in this sense they turn out to be "useless". However, since they exist, they can be used to create an additional channel with a lower rate, or by introducing a delay from several (n) allowed code combinations with "useless" symbols, form additional allowed code combinations from them, which, however, requires separate consideration.

The main difficulty in constructing the main matrix A_{n+i} is finding the adjacent matrix $B_n^{d_{min}-i}$ With an increase in n and d_{min}, this problem becomes nontrivial due to a significant increase in the enumerated options for possible realizations of the adjacent matrix $B_n^{d_{min}-i}$.

We investigate this issue with d_{min}=3

According to (15) - $K_4^3 = K_3^3 = 2$, respectively $R_4^3 = 2 \cdot R_3^3 = 2 \cdot 4 = 8$, which follows from (14). Let us consider this by an example, choosing the main matrix A_n, for which the condition d_{min} =3 and, therefore, n=3 will be fulfilled for the first time.

$$A_3 = \begin{Bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \end{Bmatrix}, \qquad (17)$$

After trivial searches, by a simple search, it is not difficult to establish the existence of adjacent matrices $B_n^{d_{min}-i} = B_3^1$, namely:

$$B_3^1 = \begin{Bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \end{Bmatrix}, \text{ or } B_3^1 = \begin{Bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \end{Bmatrix}, \text{ or } B_3^1 = \begin{Bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{Bmatrix}.$$

For further constructions, these matrices are equivalent to each other.

For definiteness, we will choose the first option and construct the main matrix according to the formula (16):

$$A_5 = \begin{Bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{Bmatrix}. \quad (18)$$

Obviously, now $K_5^3 = 4$, and from (14) it follows that $R_5^3 = R_4^3 = 8$..

Further similar reasoning allows one to construct five adjacent matrices differing from the main one by a code distance equal to 1. Namely:

$$B_5^1 = \begin{Bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{Bmatrix}, \text{ or } B_5^1 = \begin{Bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \end{Bmatrix}, \text{ or}$$

$$B_5^1 = \begin{Bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{Bmatrix}, \text{ or } B_5^1 = \begin{Bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \end{Bmatrix}, \text{ or}$$

$$B_5^1 = \begin{Bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{Bmatrix}. \quad (19)$$

Note that all 5 possible code words with weight (w=1)) and 6 of C_5^2=10 possible code words with weight (w=2) were "used" in these adjacent matrices. "Unused" code words with weight (w=2) allow constructing the remaining adjacent matrices differing from the main one by the code distance equal to 2.

$$B_5^2 = \begin{Bmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \end{Bmatrix}, \text{ или } B_5^2 = \begin{Bmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{Bmatrix}. \quad (20)$$

Further, according to formula (16), choosing for definiteness the second adjacent matrix from (20), we obtain:

$$A_6 = \begin{Bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \end{Bmatrix}. \quad (21)$$

It obviously follows from (21) that $K_6^3 = 8$, and in view of (14) - $R_6^3 = R_5^3 = R_4^3 = 8$. This means that there is one main group, six adjacent - with the minimum code distance relative to the main group equal to 1 and only one adjacent group with the minimum code distance relative to the main group equal to 2. In this case, all groups: the main and adjacent ones meet the condition d_{min}=3.

Let's indicate these groups:

$$B_6^1 = \left\{\begin{matrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 \end{matrix}\right\}, \left\{\begin{matrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{matrix}\right\}, \left\{\begin{matrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 \end{matrix}\right\},$$

$$B_6^1 = \left\{\begin{matrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 \end{matrix}\right\}, \left\{\begin{matrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{matrix}\right\}, \left\{\begin{matrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{matrix}\right\} \text{ и}$$

$$B_6^2 = \left\{\begin{matrix} 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{matrix}\right\}. \qquad (22)$$

The last adjacent group of allowed code combinations allows constructing A_7 and at the same time obtaining $K_7^3 = 16$, and $R_7^3 = R_6^3 = R_5^3 = R_4^3 = 8$.

$$A_7 = \left\{\begin{matrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \end{matrix}\right\}. \qquad (23)$$

Since the total number of groups is made up of one main group A_7 - $C_7^1 = 7$ adjacent groups B_7^1 ; then, taking into account the fact that $R_7^3 = 8$, we obtain that adjacent groups B_7^2 cannot exist. As a result, this means that for *n*=8 the number of allowed code combinations is the same as for *n*=7 i.e. $K_8^3 = K_7^3 = 16$, and from (14) it follows that the number of groups will double $R_8^3 = 2 \cdot R_7^3 = 16$. Under these conditions, using already B_7^1, one should build according to the formula (16) A_9..

As the study shows, the doubling of the number of allowed code combinations in the K_n^3 group with increasing *n* occurs if it is possible to find B_n^2. If this does not happen, then as *n* increases, the number of R_n^3 groups doubles. Note that the "growth interval" of the number of allowed code combinations $K_n^{d_{min}}$ with $d_{min} = 3$ increases as n grows as 2^{r-1}. This follows from the analysis of the increase in the number of allowed code combinations in a group with weight (*w*=3) which in $C_3^1 = 3$ cases in a code combination consisting of *n* symbols are transformed into a code combination with weight (w=2), which occurs when construction of an adjacent matrix of type B_n^1. As a result, some of the code combinations with weight (w=2), are used in matrices B_n^1, and this reduces their number, and, consequently, the possibility of constructing B_n^2.

As already noted, for the optimal Hamming code for d_{min}=3, we have:

$$n = 2^r - 1. \quad (24)$$

If we take into account that the total number of groups consists of one main group, $C_n^1 = n$ adjacent groups of type B_n^1, then for adjacent groups of type B_n^2 the following number of possibilities $L(B_n^2)$ remains:

$$L(B_n^2) = 2^r - 1 - n. \quad (25)$$

If we now substitute the value n from (24) into (25), then $L(B_n^2) = 0$, and we get a kind of "step" points when the number of allowed code combinations $K_n^{d_{min}}$ in the group does not grow, but increases number of groups $R_n^{d_{min}}$.

Summarizing the above reasoning, we can obtain an expression for $K_n^{d_{min}}$ with d_{min}=3:

$$K_n^3 = 2^{n-1-\lfloor log_2 n \rfloor}, \quad (26)$$

where rounding down to the nearest integer.

Expression (26), after simple transformations, can be reduced to (7) and also to (8). To illustrate the results obtained, we show this in Fig. 1, where the dependence k=$\log_2 K_n^3 = n- 1 - \log_2 n]$ in black, corresponding to (7).

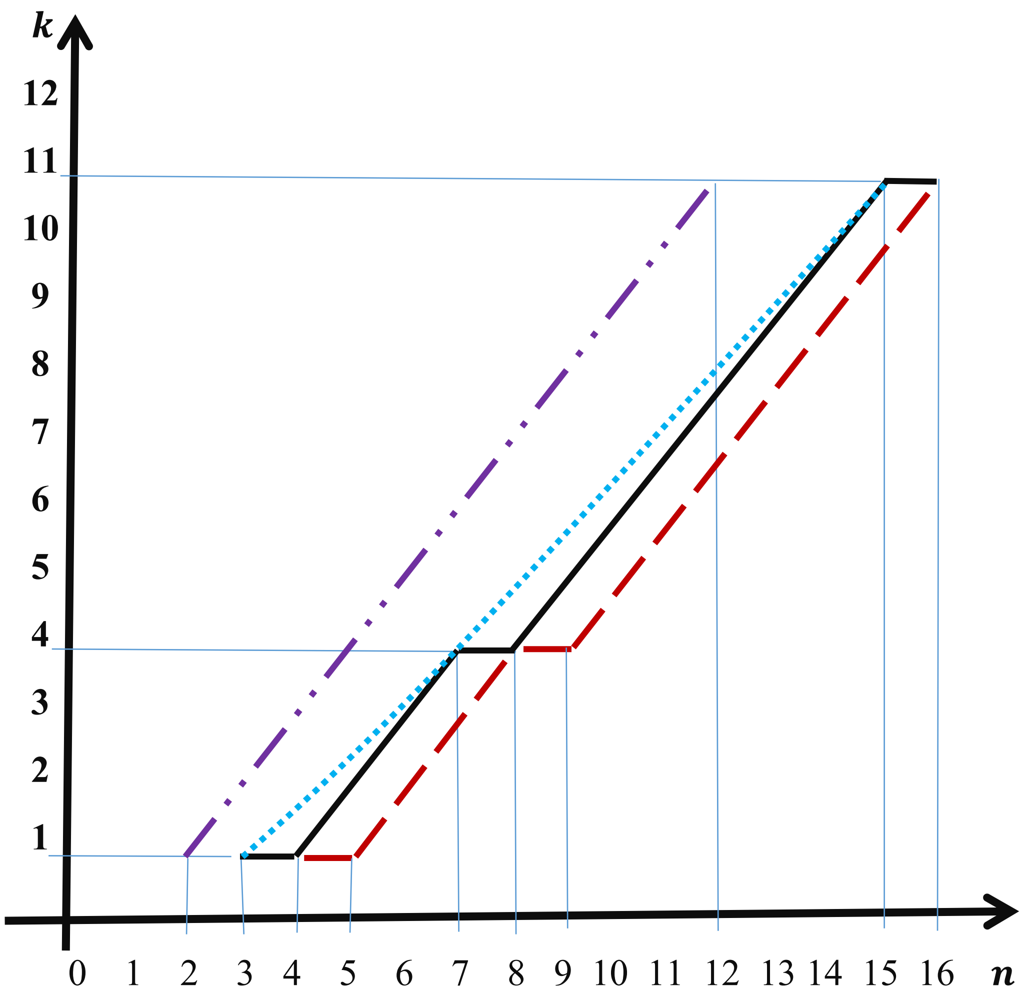

Figure 1: Dependence of ***k*** **on** ***n***

Figure 1 shows a similar dependence (in red, large dotted line) for d_{min} = 4, obtained after similar reasoning. In addition, the upper Hamming bound d_{min}= 3 (blue, dots) is also plotted here.

This upper bound exactly coincides with the values of k calculated by the exact formula (7) for n=3,15,…, i.e. for such values of n, when $\log_2 (n+1)$ given in (7) takes integer values. In other cases, there are certain differences associated with real integer solutions of expression (7), which is not taken into account in estimate (10), which gives overestimated data.

Figure 1 (purple, dash-dotted line) shows a similar dependence for d_{min}=2, which makes it possible to compare the results for an accurate estimate of the number k, which determines the number of allowed code combinations $K_n^{d_{min}} = 2^k$ in a group, depending on n for different values of d_{min}=2;3 and 4.

Continuing the research similarly to the above, by computer search of adjacent matrices, the results of the maximum achievable values of the number of information symbols k were obtained for different values of the minimum code distance $d_{min} = \overline{1,20}$ and different lengths of the code combination $n = \overline{1,34}$ are the diagrams shown in Fig. 2. (The value of k is shown on the vertical axis in Fig. 1).

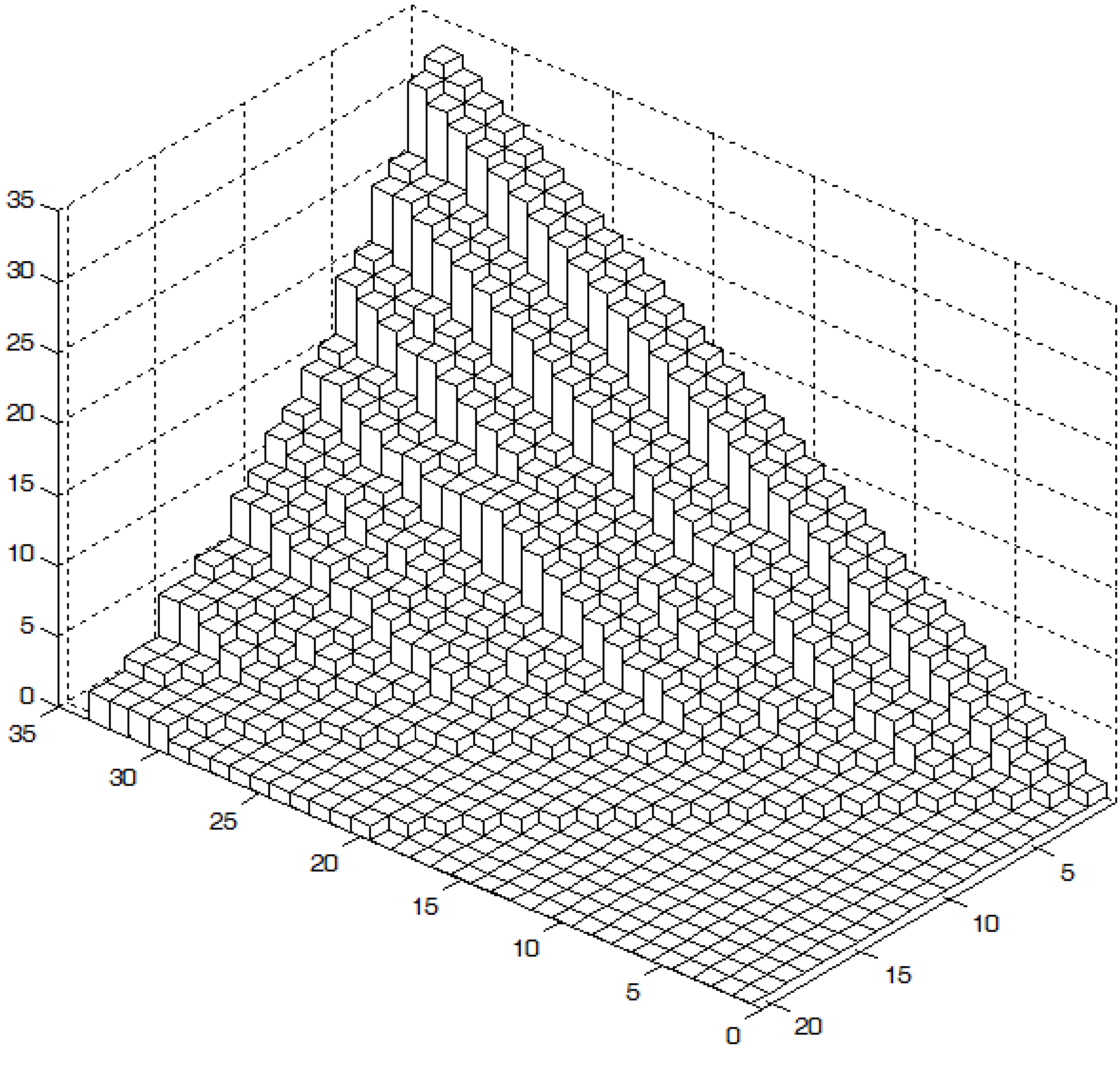

Figure 2: The number of information symbols k for different values $d_{min} = \overline{1,20}$ and $n = \overline{1,33}$

Bounds for Matrix Iterated Coding

The main idea of matrix (iterated) coding is to use constructs with several levels of error-correcting coding. As a result, the total number of information symbols is equal to the product of information symbols of each of the used error correcting codes at different coding levels. Likewise, the total length of the codeword will be equal to the product of the lengths of the error-correcting codes. In this case, the final minimum code distance increases significantly and is also equal to the product of the minimum code distances of error-correcting codes used in the design.

The most commonly used construction is with two coding levels. By choosing an error-correcting code with high efficiency at each of the levels, and thereby optimizing its choice at this level, it is not possible to guarantee that the optimization as a whole will be achieved. For example, in it was shown that choosing an optimal parity-check code with a minimum code distance equal to d_{min}=2, and introducing a two-level matrix construction, one can obtain a rapid increase in the value of the minimum code distance [8]. However, with an increase in the design levels, despite the optimization at each level separately, the total efficiency, as the ratio of the number of information symbols to the total length of the codeword k/n, decreases.

Let us investigate this issue using an example, when the construction consists of two levels of error-correcting coding and at each level a code is selected according to the Hamming bound (1). On one level, we have (n_1,k_1) code and on the other - (n_2, k_2). The total length of the codeword is n_1·n_2. In this case, the number of information symbols will be equal to $k_1 \cdot k_2$.

Based on the total length of the codeword, equal to $n_{tot}=n_1 \cdot n_2$, the number of information symbols k_{tot} can be determined according to the Hamming boundary.

Let $n_1=n_2=n$ and $k_1=k_2=k$. Then the value of the optimization efficiency as a whole in relation to optimization by parts can be calculated in relation to the Hamming boundary using the following formula:

$$E_H = \frac{k_{общ}}{k^2} = \frac{n^2 - \log_2 \sum_{i=0}^{\frac{d_{min}^2-1}{2}} C_n^i}{\left(n - \log_2 \sum_{i=0}^{\frac{d_{min}-1}{2}} C_n^i\right)^2}. \quad (27)$$

Calculations performed using formula (27) are shown in Fig. 3. (Note that only the integer values of the quantity n and d_{min})

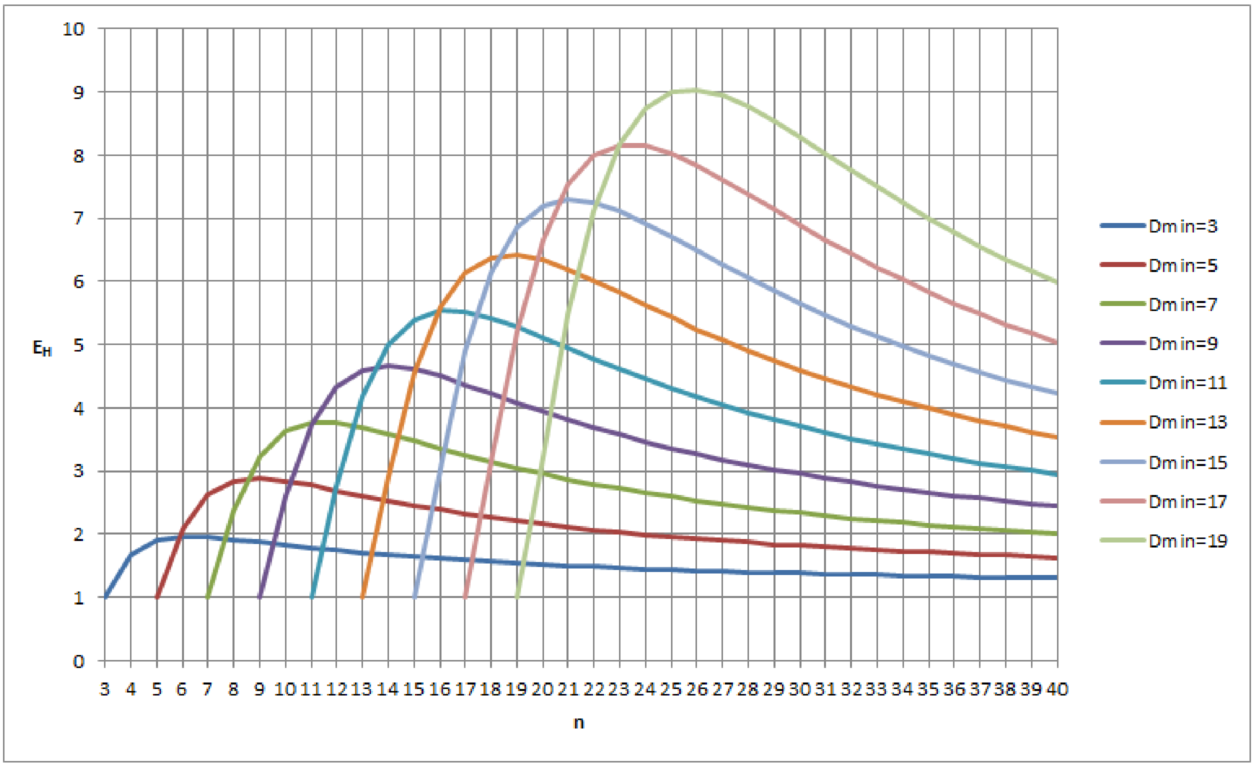

Figure 3: Efficiency of Optimization as a Whole in Relation to Optimization by Parts (Hamming Bounds)

Figure 4 shows similar dependences constructed for the Singleton boundaries (4), determined by the expression:

$$E_S = \frac{k_{общ}}{k^2} = \frac{n^2 - d_{min}^2 + 1}{(n - d_{min} + 1)^2}. \quad (28)$$

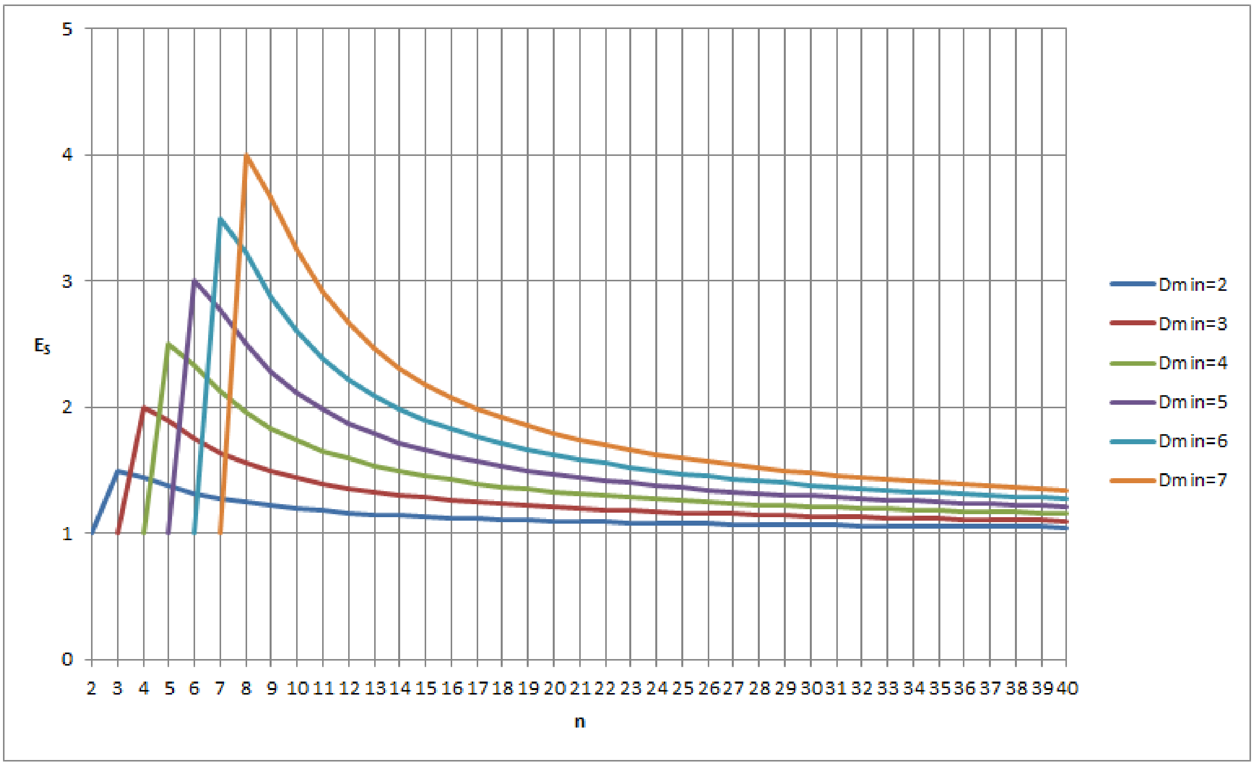

Figure 4: Efficiency of Optimization as a Whole in Relation to Optimization by Parts (Singleton Bounds)

The results of comparing the efficiency of optimization as a whole (27, 28) in comparison with optimization in parts show possible significant reserves, especially at small values of the codeword length n and large values of the minimum code distance d_{min}.

Conclusion

The obtained practically achievable boundaries as close as possible to the potential boundaries show the existing possibilities of implementing error-correcting codes with maximum efficiency. At the same time, they differ from the theoretically achievable boundaries by 5-10% and this difference cannot be reduced.

Matrix constructions of error-correcting codes, which make it possible to achieve significant correcting abilities with a rapid increase in the minimum code distance d_min and a relatively simple practical implementation, at the same time do not provide attainable limit values of efficiency as the maximum possible number of information symbols k for a given codeword length n and the value of the minimum codeword distances d_{min}.

The results of comparing the efficiency of optimization as a whole (27, 28) compared with optimization in parts show possible significant reserves, especially for small values of the codeword length n and large values of the minimum code distance d_{min}.

References

1. Peterson W., Weldon E. (1976). Error-correcting codes / Transl. from English // Ed. R.L. Dobrushin and S.I. Samoilenko. - M .: Ed. “MIR”, - 594 p.
2. Zolotaryev, V. V., & Ovechkin, G. V. (2004). Noise-resistant coding. Methods and Algorithms. Moscow, Hotline Telecom Publishing House, 126.
3. Zolotarev V.V., Zubarev Yu.B., Ovechkin G.V. (2013). Multi-threshold decoders and coding optimization theory / Ed. Academician V.K. Levin. - M .: Hot line - Telecom.- 232 p.
4. Elias, P. (1964). Encoding and decoding. the book: Lectures on the theory of communication systems. Ed. EJ Baghdadi. Per. from English M, 289-317.
5. Adzhemov A.S., Sannikov V.G. (2018). General communication theory: Textbook for universities - M .: Hot line - Telecom, 2018, 624 p.
6. Adzhemov A.S., Adzhemov S.A. (2019). Estimation of the maximum possible number of information symbols in a noise-immune block code with d_min=2,3 and 4 // Electrosvyaz, No. 11.
7. Adzhemov, A. S., & Adzhemov, S. A. (2019, March). On some features of binary code combinations. In 2019 Systems of Signals Generating and Processing in the Field of on Board Communications (pp. 1-7). IEEE.
8. Adzhemov, A. S. (2017). Possibilities of error-correcting coding when applying an iterative code. Telecommunication, (11).
9. D. Forney. (1970). Concatenated codes; Per. from English V. V. Zyablova and O. V. Popova; Ed. S. I. Samoilenko. - Moscow: Mir.-- 205 p.

10. Blokh, E. L., & Zyablov, V. V. (1982). Linear concatenated codes. Moscow, USSR: Nauka.
11. Adzhemov, A. S. (2017). Possibilities of error-correcting coding when applying an iterative code. Telecommunication, (11).
12. Berrou, C. (1993). Near Shannon limit error-correcting code and decoding: turbo code. Proc.'93 ICC.
13. Robinson, J. P., & Bernstein, A. (1967). A class of binary recurrent codes with limited error propagation. IEEE Transactions on Information Theory, 13(1), 106-113.