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## Bordism Field Theories II: Interstitial Fields and Flux Quantization

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### Abstract

In this paper, we introduce interstices on warped product manifolds, which act as discrete regulators on an otherwise smooth spacetime. We analyze flux quantization and compactification in the presence of interstitial points/loci, and prove a theorem about a lemma about manifolds arising from fibered products. In order to do so, we introduce families of fields on a compactified manifold. The co-dimension of the compactification is used in our calculations to derive boundary conditions.

Our approach contributes to the literature by advancing the mathematics of string field theory, by incorporating condensed/pyknotic and étale mathematics, with a dash of modal logic. We begin constructing the interstices using intersections of tangent and co-tangents spaces of manifolds A and B, and generalize to tangent and co-tangent groupoids. The interstices are based spaces with a fixed locus containing a non-empty subset of  $\text{Maps}(T A, T^* B)$ . Based on these topological concepts, we propose refinements to the traditional Freund-Rubin compactification model, with new corrections arising from the cobordism structure.

Finally, we construct a new partition function for functorial TQFTs that utilizes the 4-form gravitational flux, thus making this a genuine theory of quantum gravity. This work is largely speculative, but opens the door for future work on gravitational entanglement and superposition with exotic gauge fields.

**Keywords:** Interstitial Fields, TQFT, Topology, Compactification, SUGRA, Cobordisms, Supersymmetry, Families of Fields and Super-vector bundles

### Disclaimer

An appendix on errata has been added. We thank Richard Pin\_cak, as well as the editors of the Pure and Applied Mathematics Journal for helpful critique and discussion.

### Interstitial Fields

Let  $M = \mathcal{M}^{(9|2)}$  be an eleven-dimensional (pseudo-)Riemannian supermanifold with or without corners and intrinsic curvature. Suppose we have a family of fields

$$\text{Fam}_{\mathcal{M}} = \bigoplus_i^{f(m_j+n_k)} \Phi_{i \in \mathcal{I}}^{(m_j|n_k)}. (m_j + n_k \leq 11) \quad (1)$$

Varying smoothly over  $\mathcal{M}$ , where  $I \simeq Q$  is a rational subring of  $Q$ , and  $f(m_j + n_k) > 0$  is a function of the sum of odd and even dimensions. From here on out, we will let  $[m_j] = \hat{m}$  and  $[n_k] = \hat{n}$ . Following, we have two copies of a sphere; however, rather than  $S^2 \times S^2$ , we get  $S^{\hat{m}-2}$  and  $S^{\hat{n}-2}$ , and for every  $i$ , we have a bundle

$$\text{Bun}_{\Phi_i^{(m_j|n_k)}} : \mathcal{M} \longrightarrow (S^{\hat{m}-2} \times S^{\hat{n}-2})^{\text{codim } S^{\min(\hat{m}, \hat{n}-2)}} \quad (2)$$

And the *generic fiber* of this bundle winds around the first sphere  $\hat{m}$  times and the second  $\hat{n}$  times [1]. Again, this extends to a family of its own:

$$\text{Fam}_{\text{Bun}_{\Phi_i^{(m_j|n_k)}}} : \text{Fam}_{\mathcal{M}} \longrightarrow \bigoplus_{\hat{m}+\hat{n}}^{f(9+2)} (S^{\hat{m}-2} \times S^{\hat{n}-2})^{\text{codim } S^{\min(\hat{m}, \hat{n}-2)}} \quad (3)$$

Where again the function is strictly increasing.

### Notation 1

We shall use  $\tilde{\times}$  to denote a fibered product; i.e.,  $\mathbf{A} \tilde{\times} \mathbf{B} = \mathbf{A} \times_{\sqsubset} \mathbf{B}$ , where  $\sqsubset : \text{Fib}(\mathbf{A}, \mathbf{B})$  is the fiber and  $\text{Fib}(\mathbf{A}, \mathbf{B})$  is the category of fibrations  $\pi(\mathbf{A}) \rightarrow \pi(\mathbf{B})$ .

This notation is introduced, because  $S^{\hat{m}-2} \tilde{\times} S^{\hat{n}-2}$  is actually an *Einstein or warped product* manifold is chosen with respect to a *special point*  $\wp$  lying in  $(T_x(\mathbf{A}) \cap T_y^*(\mathbf{B})) \cup (T_{x'}^*(\mathbf{A}) \cup T_{y'}(\mathbf{B}))$ , which we shall call the interstice of A and B. We call the special point an interstitial point if an action is conserved with respect to  $\wp$ , or colloquially, if there is a local mirror symmetry around the special point.

### Definition 1

The interstice of two groupoids A and B is the product

$$\mathcal{T}\mathbf{A} \times_{\text{Fib}(\mathbf{A}, \mathbf{B})} \mathcal{T}^*\mathbf{B} \quad (4)$$

Where  $\tilde{\times}$  is a 2-categorical product.

### Notation 2

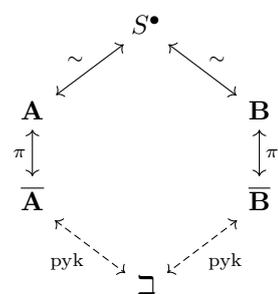
We shall denote the interstice of two groupoids by  $\text{inter}_{\mathbf{A}, \mathbf{B}}$ .

### Lemma 1

If  $\mathbf{A} \simeq \mathbf{B}$  via weak equivalence, then  $\pi(\text{Fib}(\mathbf{A}, \mathbf{B}))$  is étale.

Proof. First, it must be shown that  $\pi(\text{Fib}(\mathbf{A}, \mathbf{B}))$  is étale. First, note that due to the spacegroupoid equivalence, A and B are both groupoids. Since the weak equivalence is assumed, we have  $B = \mathcal{F}([0,1]) \times A$ . Since the unit interval is contractable, then any homeomorphism must also be so. Thus, the contractability of B depends on that of A; i.e., if T A is smooth with respect to infinitesimal deformation, then  $T^* \mathbf{B}$  will also be smooth, and thus so too will the interstice be.

### We obtain the following diagram



And since  $\sqsubset \ni \wp$  is pyknotic, it condenses to  $\wp$  in the Zariski topology. Notice that every morphism in the above diagram is invertible; therefore, the category forms a one-object subcategory of  $\mathcal{Gpd}\acute{E}t$ , and each of the objects are 1-small subobjects.

### Remark 1

See Appendix A for more details on this proof.

## BFT

In BF field theories, the integration measure, which is typically given by a  $E = d^d x$  term, is replaced by the action:

$$S_{BF} = \text{Tr}(B \wedge F) \quad (5)$$

Where  $B$  is a  $(d - 2)$ -form valued in the adjoint representation of a gauge group  $G$ ,

$$F = dA + A \wedge A$$

Is the curvature 2-form on a gauge field  $A$ , and the integral is taken over a generic  $d$ -dimensional spacetime.

When we superize this theory<sup>1</sup>, instead of the normal  $d^d x$  term, we make the replacement

$$E = d^{\hat{m}} x_{\mu} + d^{\hat{n}} x_{\nu}$$

Where the left-hand term is the even action and the right-hand term is the odd action, and the odd action acts a Lagrangian multiplier or "boost" to the even one. However, if we have

$$d^{\hat{m}} x_{\mu} + d^{\hat{n}} x_{\nu} = 0$$

We are left with an anomaly.

If  $\square/\mathcal{C}$  is a context<sup>2</sup> over a category  $\square$  and  $s \in \square \simeq \hat{b} \in \mathbb{L}^{\hat{m}|\hat{n}}$  is an object in  $\square$ , then the anomaly means that for a measurement  $q(E)$  of  $E$ , we get

$$q(E) = \nu_E \iff d^{\hat{m}} x_{\mu} + d^{\hat{n}} x_{\nu} = 0$$

Where  $\nu_E \sim \emptyset$  is Emerson's element<sup>3</sup>. Thus, the anomalous wave will not be observed or detected by any measuring device.

To cancel the anomaly, first consider a 2-categorical BF theory, where rather than merely having a gauge field alongside a Lagrangian multiplier, we promote these to higher categorical objects. We extend the BF action to interstices:

$$S_{\text{int}} = \int_{\mathcal{M}} \text{Tr}(B \wedge F) + \int_{\Sigma=\text{inter}_{A,B}} \text{Tr}(C \wedge R) \quad (6)$$

Where:

- $B$  is a  $(d - 2)$ -form, as in ordinary bulk BFT
- $C$  is a new  $(d - 3)$ -form field supported only on the interstice and
- $R$  is the curvature of the connection in a higher groupoid, encoding how flux propagates between the two regions and  $R$  describes how the gauge field  $A$  "jumps" across  $\Sigma$ .

The action of a bulk BFT is topological, and possesses a gauge invariance:

$$A \longrightarrow A + d\lambda \quad B \longrightarrow B + d\Lambda$$

Where  $\lambda$  is a local gauge transformation parameter governing the infinitesimal deformation of individual groupoid fibers and  $\Lambda$  is a higher order one, controlling the deformation of the entire fiber structure in a manner compatible with étale descent. They satisfy an inverse limit condition:

$$\Lambda = \varprojlim \lambda_i$$

Where the  $\lambda_i$  are local gauge parameters for the individual étale covers indexed by  $i$ , and  $\Lambda$  is the total gauge freedom emerging from condensation.

## Interstice as a Defect Theory

The interstice is a codimension-1 hypersurface  $\Sigma$ ; it can serve as a boundary condition of sorts, and it is embedded as:

$$\Sigma = M_+ \sqcup M_-$$

Where  $M_+$  and  $M_-$  are two adjacent regions. The field  $C$ , which is defined locally on  $\Sigma$  mediate between these two regions, acting as a sort of regulator which controls the resolution of observation.

As the action  $S_{\text{int}}$  varies, so too should there be a corresponding variation in the anomaly inflow equation:

$$S_{\text{inflow}} = \frac{1}{2}(q(E) + q(\text{Fam}_{\mathcal{M}})) \quad (7)$$

Rather than the usual gauge group  $G$ , we consider 2-groups or Lie groupoids  $R$ , replacing  $A$  by  $A + \Gamma$  where  $\Gamma$  is a 1-form connection on a 2-group bundle. We define  $R$  by:

$$R = F_+ - F_-$$

Let's assume we have a principle  $G$ -bundle over the bulk  $M$  with a gauge connection  $A$ , leading to the field strength:

$$F = dA + A \wedge A$$

On the interstice, the gauge field is not necessarily smooth, leading to a jump condition:

$$F_+ - F_- = dR + A \wedge R + R \wedge A$$

Under a gauge transformation

$$A \rightarrow A + d\lambda + [A, \lambda]$$

, the interstitial field  $C$  transforms as

$$R \rightarrow R + [R, \lambda]$$

To directly compute the flux, we use the equation:

$$\Phi = \int_{\Sigma} R$$

Which we are safe to re-write as

$$\Phi = \int_{\Sigma} (F_+ - F_-)$$

and if  $G = dC + C \wedge C$ , then by Stoke's theorem we get:

$$\Phi = \int_{\partial\Sigma} C$$

## Refining the Freund-Rubin Ansatz for Bordism Field Theories

The usual Freund-Rubin compactification assumes the following ansatz for an 11-dimensional SUGRA metric [1]:

$$ds^2 = ds_{\text{AdS}_d}^2 + L^2 ds_X^2$$

Where:

- $\text{AdS}_d$  is a  $d$ -dimensional anti-de Sitter space.
- $X$  is a compact internal space, traditionally assumed to be  $S^{k \leq 9}$ .
- $L$  is the characteristic length scale determined by the fluxes.

Here, we shall modify the Freund-Rubin four-form field strength,  $F_4$ , which is

$$F_{\mu\nu\rho\sigma} \sim \Lambda \epsilon_{\mu\nu\rho\sigma}$$

with  $\Lambda$  proportional to the vacuum energy and  $\epsilon_{\mu\nu\rho\sigma}$  the volume form of  $\text{AdS}_d$ , while the rest of the components of  $F_4$  vanish except possibly along the compact space  $X$ . Following in the spirit of and in this document, we define the following warping function [2]:

$$f(m_j + n_k) = \int_{\text{Fam}_{\Sigma}} \omega_{\mathcal{M}, \mathcal{N}} \quad (8)$$

Where  $\text{Fam}_\Sigma$  is the interstitial bordism family over compact spaces, and  $\omega_{M,N}$  is a differential form encoding the topological transition data between two regions.

### Flux Quantization

Since  $F_4$  satisfies

$$dF_4 = 0 \quad d \star F_4 \neq 0$$

the flux is constrained by the topology. However, in the presence of interstices, the jump condition

$$F_4^{(\text{Bulk})} - F_4^{(\text{Interstice})} = dC$$

Which suggests that the bordism family structure modifies flux quantization via:

$$\int_{\text{Fam}_M} F_4 = \sum_i \int_{X_i} F_{4,i} \neq 0 \quad (9)$$

Furthermore, the four-form traditionally satisfies [3]:

$$\int_{X_4} F_4 \in 2\pi\mathbb{Z} \quad (10)$$

Which becomes

$$\int_{X_4} F_4 \in 2\pi(\mathcal{I} \simeq \mathbb{Q}) \quad (11)$$

The flux is no longer a single conserved quantity, but is distributed across a family of compactifications:

$$\text{Fam}_{F_4} = \bigoplus_i^{f(m_j+n_k)} F_{4,i}^{(m_j|n_k)}$$

For a given bordism family  $\text{Fam}_M$ , flux quantization must hold on the entire family, leading to:

$$\sum_i \int_{X_{4,i}} F_4 = \int_{\text{Fam}_M} F_4 \in 2\pi\mathbb{Q} \quad (12)$$

and, by letting  $X_{4,+}$  denote the compact space on one side of  $\Sigma$  and  $X_{4,-}$  the right-hand compact space, we get

$$\int_{X_{4,+}} F_{4,+} - \int_{X_{4,-}} F_{4,-} = \int_{\Sigma} dC \quad (13)$$

This suggests that flux can discontinuously jump across the interstice, leading to a generalized conservation law where  $dC$  represents anomaly inflow at the interstice.

### Impact of Cobordism Geometry on the Freund–Rubin Ansatz

The Freund–Rubin ansatz posits a background of the form  $M^p \times X^q$ , where  $M^p$  is a  $p$ -dimensional Lorentzian spacetime and  $X^q$  is a compact Riemannian manifold, with  $p + q = D$ , the total spacetime dimension. A  $(q + 1)$ -form field strength  $F_{q+1}$  is assumed to thread the compact space, supporting a warped or unwarped product geometry depending on the theory. Explicitly, the field strength satisfies

$$F_{q+1} = f \text{vol}_X,$$

Where  $\text{vol}_X$  is the volume form on  $X^q$ , and  $f$  is a constant proportional to the flux. The Einstein equations then require  $M^p$  and  $X^q$  to be Einstein manifolds with curvatures of opposite sign, dictated by the flux.

When the internal space  $X^q$  is replaced by or embedded into a cobordism  $W^{q+1}$  such that  $\partial W = X_1^q \sqcup X_2^q$ , the geometry and topology of the full configuration become subtler. The topological class of  $W$  can influence the admissibility of fluxes and the consistency of the higherdimensional theory, especially in the context of flux quantization and anomaly inflow.

From a differential form perspective, the flux must now satisfy global cohomological constraints. The closed form  $F_{q+1}$ , being sourced on  $X^q$ , extends to a cocycle in  $H^{q+1}(W, \mathbb{Z})$ , subject to the condition

$$\int_{\Sigma^{q+1}} F_{q+1} \in 2\pi\mathbb{Z}$$

for all closed  $(q + 1)$ -cycles  $\Sigma^{q+1} \subset W^{q+1}$  consistent with Dirac quantization. If  $W$  is nontrivial in bordism (i.e.,  $[X_1] \neq [X_2]$  in  $\Omega_q$ ), then the global consistency of  $F_{q+1}$  may require the inclusion of boundary terms in the action or additional topological data, such as differential cocycles or Wu structures.

Furthermore, the variation of the Freund–Rubin flux across the cobordism can affect the effective potential in the dimensionally reduced theory. In compactifications where the flux varies smoothly across  $W$ , the effective four-dimensional potential  $V(\phi)$  (where  $\phi$  parametrizes moduli fields such as the volume of  $X$ ) may acquire corrections of the form:

$$V(\phi) \sim \frac{f^2(\phi)}{\text{Vol}(X(\phi))^2} + \dots,$$

With  $f(\phi)$  modulated by topological data of  $W$ . In scenarios where  $W$  supports a higher structure (e.g., string cobordism or oriented cobordism with additional geometric structure), the Freund–Rubin mechanism may even be obstructed or enriched by secondary invariants (e.g., the eta invariant or index densities).

Thus, incorporating cobordism geometry into the Freund–Rubin framework allows for a refined classification of flux backgrounds and compactifications, extending the traditional ansatz into a richer topological landscape. This has implications not only for the existence and stability of Freund–Rubin vacua but also for their compatibility with anomaly constraints in string and Mtheory.

### Bordism Correction Terms

The bordism group  $\Omega_n^U$  contains torsion components, meaning that certain bordism classes do not bound a manifold but contribute discrete factors. This implies downstream constraints in, e.g., flux quantization.

Consider the long exact sequence in homology associated with a bordism  $W$ :

$$\dots \rightarrow H_{n+1}(W) \rightarrow H_n(\partial W) \rightarrow H_n(W) \rightarrow H_{n-1}(\partial W) \rightarrow \dots$$

Whence  $H_n$  has torsion, then we must modify the flux quantization condition by appending:

$$\int_{\Upsilon_{\text{out},1}} F_4 - \int_{\Upsilon_{\text{out},2}} = \int_{\Sigma} dB + \text{Tor}(\Omega_*^U) \quad (14)$$

Where each of the  $\Upsilon_{\text{out},*}$  are outputs of the bordism,  $W : \Upsilon_{\text{in},0} \rightrightarrows \Upsilon_{\text{out},1} \times \Upsilon_{\text{out},2}$  is the bordism itself, and for every submanifold  $\Upsilon_{\bullet,*}$ , there is an associated Hilbert space  $\mathcal{H}_{\Upsilon_{\bullet,*}}$  so that the bordism is a bifurcation of a unitary state living on  $\mathcal{H}_{\Upsilon_{\text{in},0}}$ . The output manifolds represent possible states<sup>4</sup>, which may be considered as entangled or in superposition.

### Remark 2

The Hilbert space splits as a pair of maps context slices over a spacetime groupoid:

$$\mathcal{M}_{/\square} \begin{matrix} \xrightarrow{f} \\ \xrightarrow{g} \end{matrix} \mathcal{M}_{/\diamond_1} \times \mathcal{M}_{/\diamond_2}$$

Where the collection of interstitial loci<sup>5</sup>  $[\square]$  :  $\square$  may or may not be preserved, depending upon the severity of dissipation across time slices. We could, in principle, extend this to arbitrarily many future states by constructing a family

$$\text{Fam}_{\mathcal{M}_{/\diamond_{>0}}} = \text{Hom}(\mathcal{M}_{/\square}, \mathcal{M}_{/\diamond_{>0}})$$

and by letting

$$\square x = \diamond^0 x = \diamond_0$$

for all “packets”  $x \in \mathcal{M}$ .

To superize, we let  $f \sim g$  for every function  $(g \neq f) \in \text{Hom}(\mathcal{M}_{/\square}, \mathcal{M}_{/\diamond_{>0}})$  and take

$$\text{deg}[f] = (-1)^{\text{deg } \square + \text{deg } \diamond} > 0$$

where  $\text{deg}$  is the Fredholm index. Then, if  $x$  is a quantum, its image under any  $\tilde{f} \in [f]$  is a "squantum," or superpartner.

Where  $\text{deg}$  is the Fredholm index. Then, if  $x$  is a quantum, its image under any  $f_e \in [f]$  is a "squantum," or superpartner.

### Explicit Computations

The complex bordism groups  $\Omega_n^U$  are given by the homotopy groups of the Thom spectrum MU

$$\Omega_*^U = \pi_* MU$$

For low dimensions, the groups contain torsion elements; in particular:

$$\Omega_2^U = 0 \quad \Omega_4^U = \mathbb{Z} \quad \Omega_6^U = 0 \quad \Omega_8^U = \mathbb{Z}/2$$

The  $\mathbb{Z}/2$  factor in  $\Omega_8^U$  means that there exists an 8-manifold  $\mathcal{M}^8$  which represents a non-trivial bordism class of order 2; that is, this manifold bounds a complex 9-manifold twice. Thus, we modify (3.1) appropriately:

$$\int_{\Sigma_{\text{interstice}}} dB \equiv 0 \pmod{2} \quad (15)$$

A well-known generator of the torsion in  $\Omega_8^U$  is the complex projective space  $CP^4$  with a specific framing. The fundamental fact is that  $2CP^4$  is bordant to zero in  $\Omega_8^U$ , meaning that twice this manifold bounds a 9-dimensional complex-oriented manifold.

Thus, we take

$$\mathcal{M}^8 \cong CP^4$$

which carries a natural complex structure, with tangent bundle related to tautological line bundle  $\mathcal{O}(-1)$  over  $CP^4$ . Since

$$H^4(CP^4) = \mathbb{Z}$$

a flux  $F_4$  on this space satisfies the quantization condition

$$\int_{CP^4} F_4 \in \mathbb{Z}$$

However, if  $CP^4$  represents a torsion class in bordism, then twice this flux must trivialize in some 9-dimensional bounding space, which introduces a mod 2 quantization condition. So, we kick the can down the road a bit further, and modify (3.2) to become

$$\int_{\Upsilon_{\text{out},1}} F_4 - \int_{\Upsilon_{\text{out},2}} F_4 = \int_{\Sigma} dB + k \pmod{2} \quad (16)$$

thus constraining the spectrum of allowed flux values on the interstice.

### Modified Partition Function

A quantum path integral must sum over all possible flux configuration, including the twisted sector. A good partition function would be:

$$\mathcal{Z} = \sum_{[F_4]} e^{iS[F_4]} \cdot e^{i\Theta([F_4])}$$

Where  $S[F_4]$  is the standard action term and  $\Theta([F_4])$  is a torsion-dependent quantum phase factor that introduces interference between topologically distinct sectors.

## Pyknotic Objects

### Recollection 1

A pyknotic structure means that objects carry a pro-finite topology, enabling a sort of inverse limit topology:

$$A \leftarrow A_1 \leftarrow \dots \leftarrow A_2 \dots$$

where each of the  $A_i$  are progressively higher resolution étale covers;  $A$  then emerges as the limiting object:

$$A = \varprojlim A_i$$

The limit  $\varprojlim A_i$  is naturally a stack over the étale site, which means that:

$$H_{\text{ét}}^n(A) = \varprojlim H_{\text{ét}}^n(A_i)$$

### Remark 3

The above proof relies on the work of Barwick and Haine [4]. The condensation map  $\text{pyk}$  ensures that the category collapses to a smallest étale subcategory, effectively selecting the most refined structure. Since Zariski sheaves are already local in the étale topology, the condensation essentially restricts us to the “correct” structure.

1-smallness follows from:

### Proposition 1

The limit is well-defined in the étale topology, remains a 1-small subobject, and respects étale descent. Hence, condensation ensures that  $\varprojlim A_i$  stays fully faithful and remains in  $\text{GpdEt}$ .

Proof. For any two objects  $x$  and  $y$  in the inverse system, the mapping space:

$$\text{Hom}_{\varprojlim A_i}(x, y) \qquad \varprojlim \text{Hom}_{A_i}(x_i, y_i)$$

may be computed as a limit of the corresponding Hom-sets at each stage:

$$\varprojlim \text{Hom}_{A_i}(x_i, y_i)$$

Since each  $A_i$  is étale, its morphisms are locally homeomorphic to a space over a base. That is, for every object  $x_i$ , there is a small neighborhood  $U_i$  such that  $\pi : U_i \rightarrow S^*$  is a local diffeomorphism. This means every morphism lifts uniquely in the étale topology, and there are no extra degrees of freedom in the inverse system.

The condensation property of  $\sqsupset$  ensures that at the limit, the étale transition maps glue uniquely; that is, there are no trivial kernels in our inverse limit. This is because pyknosis enforces Zariskilocal triviality of the interstitial morphisms, meaning that all the lifting maps remain étale.  $\text{Et}^* \text{Gpd}$  behaves as a single atomic object, and the proposition is shown to be true.

## Errata

### Lemma 1

In the étale topology, smoothness must be established via local properties, such as the existence of atlases or effective descent in the context of Lie groupoids.

## Higher Gauge Structures and Descent

Gauge transformations are usually local and glue over topological or étale covers via cocycle data. Let  $\{\lambda_i\}_{i \in I}$  be a collection of local gauge transformations defined over a cover  $\{U_i\}$  of a base space  $X$ , with compatibility data  $\lambda_{ij}$  on overlaps  $U_i \cap U_j$ . A global gauge transformation  $\Lambda$  arises via descent if the stack of fields satisfies effective descent. We avoid interpreting unless working in a category where pro-objects or formal completions are defined.

## Bordism

The correct structure of the complex bordism group in dimension 8 is:

$$\Omega_8^U \cong \mathbb{Z} \oplus \mathbb{Z}/24$$

This replaces the earlier incorrect claim of  $\mathbb{Z}/2\mathbb{Z}$ . The presence of a free part and a 24-torsion component has implications for both cobordism classification and physical applications, such as anomaly cancellation and quantization.

Given the above correction, we replace

$$\int_{\Sigma} dB \cong 0 \pmod{2}$$

by

$$\int_{\Sigma} dB \cong 0 \pmod{24}$$

### Relevance to Sugra

Latly, we address concerns of relevance to 11-dimensional supergravity. The interstices are interpreted as codimension-1 topological defects or domain walls. Their interaction with the equations of motion in 11D supergravity must be made explicit [5-11]. The field equation is

$$d * G = \frac{1}{2} G \wedge G$$

and, to be physically consistent, the interstices should either

- source  $G$ , as in brane-like objects (e.g.  $M9$ -branes).
- mediate transitions between different flux vacua, like phase interfaces.

We propose modeling them via jump conditions or cobordism data between field configurations, though a detailed treatment is left for future work.

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