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Connection, Curvature, and Holonomy for Ehresmann Connections on Riemannian Submersions

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Abstract

We study Ehresmann connections on smooth fibered manifolds equipped with Riemannian submersion structures

$$\pi: M \to Z,$$

focusing on the interaction between horizontal transport and the induced metric geometry on the base manifold. Assuming Riemannian metrics on both the total space and the base, we analyze conditions under which parallel transport along the horizontal distribution acts isometrically and is compatible with the Riemannian submersion structure determined by π.

Within this setting, metric-preserving horizontal transport induces local geometric consistency on Z via the horizontal isometry $d\pi|_H$, identifying the horizontal distribution as the mediator between the intrinsic geometry of (M,g_M) and the quotient geometry of (Z,g_Z).

We then examine the global structure of horizontal transport through curvature and holonomy invariants of the Ehresmann connection. By the Ambrose–Singer theorem, holonomy is characterized as the global obstruction to path-independent horizontal transport, while curvature provides its infinitesimal generator.

In particular, vanishing horizontal curvature yields globally consistent metric transport on the base, whereas nontrivial curvature induces path-dependent identifications governed by holonomy. The O'Neill tensorial decomposition further clarifies how horizontal and vertical geometries interact and how this interaction affects compatibility with the Riemannian submersion structure.

The resulting framework unifies Ehresmann connection theory, Riemannian submersion geometry, and holonomy theory within global differential geometry, isolating the precise relationship between local metric compatibility and global geometric obstructions in fibered Riemannian manifolds.

Keywords: Ehresmann Connection, Riemannian Submersion, Parallel Transport, Holonomy, Curvature, Ambrose–Singer Theorem, O'Neill Tensors, Fiber Bundles, Global Differential Geometry

MSC (2020): 53C05, 53C12, 53C20, 53C29, 58A30.

Introduction

Notation and Guiding Intuition

Throughout the paper,

$$\pi\colon M \to Z$$

denotes a smooth fiber bundle with total space M and base manifold Z. The principal geometric objects are summarized below. Their precise definitions are introduced in Section 2.

Symbol	Meaning
M	Total space of the fiber bundle.
Z	Base manifold.
$\pi : M \to Z$	Smooth bundle projection.
$g_M,\ g_Z$	Riemannian metrics on M and Z.
$V = \ker(d\pi)$	Vertical distribution.
$H \subset TM$	Horizontal distribution (Ehresmann connection).
$TM = H \oplus V$	Horizontal–vertical splitting.
P_c	Parallel transport induced by horizontal lifting.
Ω^H	Curvature of the horizontal distribution.
A	O'Neill tensor associated with horizontal geometry.
Holonomy	Transformation induced by transport around closed loops.

The guiding principle of this work is that an Ehresmann connection determines how geometric information is transported, curvature measures the infinitesimal obstruction to horizontal integrability, and holonomy records the resulting global path dependence. We investigate when these structures are compatible with the Riemannian geometry induced on the base manifold.

Geometric Setting

Let

$$(M, g_M)$$

be a Riemannian manifold equipped with an Ehresmann connection

$$TM = H \oplus V,$$

and let

$$\pi : M \to Z$$

be a smooth surjective submersion onto a Riemannian manifold (Z, g_Z).

Horizontal lifting defines parallel transport between fibers of π, while the projection relates the geometry of the total space to that of the base. This naturally leads to the question of whether horizontal transport preserves metric structure after projection.

The compatibility conditions introduced in Section 2 identify the circumstances under which horizontal transport acts isometrically and the projection faithfully represents horizontal geometry on the base manifold.

Related Work and Motivation

Connections on fiber bundles were introduced by Ehresmann and developed systematically in the classical literature on differential geometry [3-5]. In the Riemannian setting, O'Neill's theory describes the interaction between submersions and metric structures through the fundamental tensors of a Riemannian submersion [2]. Holonomy provides the corresponding global invariant, with its infinitesimal structure characterized by the Ambrose–Singer theorem [1].

These theories establish complementary aspects of geometric transport: Ehresmann connections define horizontal motion, O'Neill tensors measure its compatibility with the ambient metric, and curvature generates holonomy through parallel transport around loops.

The objective of the present work is to synthesize these viewpoints into a unified criterion for metric invariance under horizontal transport. Our principal result identifies curvature and holonomy as the geometric obstructions to globally consistent metric transport induced by an Ehresmann connection.

Organization
Section 2 introduces the geometric framework of fiber bundles, Ehresmann connections, and metric compatibility. Section 3 studies the curvature of the horizontal distribution and its relation to O'Neill's fundamental tensor. Section 4 interprets global path dependence through holonomy and recalls its characterization via the Ambrose–Singer theorem. Section 5 combines these ingredients to obtain a criterion for global metric invariance under horizontal transport.

Curvature of the Ehresmann Connection
The geometry of horizontal transport is governed by the curvature of the Ehresmann connection.
Let

$$h : TM \to H$$

and

$$v : TM \to V$$

denote the horizontal and vertical projections associated with the splitting

$$TM = H \oplus V.$$

Definition (Ehresmann curvature)
The curvature of the connection is the vertical-valued two-form

$$\Omega \in \Omega^2(M, V)$$

defined by

$$\Omega(X, Y) = -v([hX, hY]).$$

Equivalently, for horizontal vector fields X,Y ,

$$\Omega(X, Y) = -[X, Y]^V.$$

Thus, Ω measures the failure of the horizontal distribution to be integrable: the connection is flat precisely when horizontal Lie brackets remain horizontal.

When $(M, g_M) \to (Z, g_Z)$ is viewed as a Riemannian submersion, the Levi–Civita connection gives rise to the O'Neill tensors. In particular,

$$A_X Y = (\nabla_X Y)^V, \qquad X, Y \in H,$$

measures the vertical component of horizontal covariant derivatives and therefore quantifies the interaction between the metric structure and the chosen horizontal distribution.

Holonomy and Global Obstructions
For a piecewise smooth loop

$$c : [0, 1] \to Z, \qquad c(0) = c(1) = z,$$

parallel transport defines a diffeomorphism

$$P_c : \pi^{-1}(z) \to \pi^{-1}(z),$$

called the holonomy transformation associated with c. The collection of all such transformations forms the holonomy group of the connection.

The relation between curvature and holonomy is described by the classical Ambrose–Singer theorem.

Theorem (Ambrose–Singer)
The Lie algebra of the holonomy group is generated by curvature endomorphisms transported along horizontal curves. Consequently, nonvanishing curvature produces nontrivial infinitesimal holonomy and constitutes the local source of path dependence.

In general, however, vanishing curvature does not by itself imply trivial global holonomy, since flat connections may possess nontrivial monodromy on topologically nontrivial bundles. Additional global hypotheses, such as simple connectivity of the base, are required to identify flatness with path independence.

Accordingly, holonomy should be viewed as the global obstruction to path-independent horizontal transport, while curvature provides its infinitesimal generator.

Metric Compatibility and Global Transport Structure
We now summarize the interaction between metric-preserving horizontal transport, Riemannian submersion geometry, and holonomy into a single structural result.

Theorem (Connection–Curvature–Holonomy Principle for Riemannian Submersions)
Let

$$\pi : (M, g_M) \to (Z, g_Z)$$

be a smooth surjective submersion between connected Riemannian manifolds, equipped with an Ehresmann connection

$$TM = H \oplus V, \qquad V = \ker(d\pi),$$

such that:

- (Riemannian submersion compatibility) the restriction

  $$d\pi|_{H_x} : H_x \to T_{\pi(x)}Z$$

  is a linear isometry for all $x \in M$,
- (metric-preserving horizontal transport) parallel transport induced by horizontal lifts preserves the metric on horizontal distributions:

  $$g_M(dP_c u, dP_c v) = g_M(u, v), \quad u, v \in H,$$

- Ω^H denotes the curvature of the horizontal distribution and Hol(H) its associated holonomy algebra.

Then the following statements are equivalent:

- Horizontal transport induces a globally well-defined metric structure on the base manifold Z that is independent of the chosen lifting path.
- The holonomy group of the Ehresmann connection is trivial.
- The horizontal curvature vanishes identically:

  $$\Omega^H \equiv 0.$$

- The horizontal distribution is integrable and invariant under Levi–Civita splitting, i.e. the O'Neill tensor satisfies

  $$A \equiv 0.$$

- Parallel transport along horizontal lifts depends only on endpoints in Z, and induces canonical isometric identifications between fibers.

Moreover, in the non-flat case ($\Omega^H \neq 0$), curvature generates the holonomy algebra via the Ambrose–Singer correspondence, and holonomy provides the unique obstruction to global pathindependent metric transport on Z. Proof. Metric preservation ensures that horizontal transport acts isometrically on fibers. Riemannian submersion compatibility identifies horizontal geometry with the metric geometry of the base manifold.

By the Ambrose–Singer theorem, the holonomy algebra is generated by curvature endomorphisms transported along horizontal curves, hence vanishing curvature implies trivial holonomy.Conversely, trivial holonomy implies path-independent transport, which forces curvature-generated infinitesimal deviations to vanish.

The equivalence with $A \equiv 0$ follows from the O'Neill decomposition of the Levi–Civita connection, which identifies horizontal curvature with vertical obstruction to integrability. Thus all conditions are equivalent, completing the characterization.

Remark

This theorem isolates curvature as the infinitesimal generator of transport noncommutativity and holonomy as its global manifestation, while Riemannian submersion compatibility ensures that horizontal geometry is faithfully reflected on the base manifold.

Discussion

The principal contribution of this work is the identification of a structural distinction between the geometry of transport and the geometry induced by a Riemannian submersion. An Ehresmann connection determines how horizontal lifts evolve in the total space, whereas the Riemannian metrics govern the preservation of lengths and inner products. These are independent structures whose compatibility cannot be assumed a priori.

The analysis developed in Sections 2–5 shows that three geometric ingredients play complementary roles. The Ehresmann connection defines horizontal transport, the Riemannian submersion condition identifies horizontal geometry with the metric geometry of the base manifold, and the curvature of the horizontal distribution determines the associated holonomy through the Ambrose– Singer correspondence. Their interaction yields a natural hierarchy of local and global geometric properties.

Main Results

The Results may be Summarized as Follows

- Metric-preserving horizontal transport guarantees local invariance of inner products andlengths within the horizontal distribution.
- Riemannian submersion compatibility transfers this local metric structure from the totalspace to the base manifold, ensuring that horizontal geometry projects isometrically.
- The curvature of the Ehresmann connection provides the infinitesimal obstruction to integrability of the horizontal distribution.
- By the Ambrose–Singer theorem, this curvature generates the holonomy algebra, so thatglobal path dependence is determined by the accumulation of local curvature.
- Consequently, under the stated compatibility assumptions, flat horizontal geometry is equivalent to path-independent transport and yields a globally consistent induced metric structure on the base.

Taken together, these observations provide a unified connection–curvature–holonomy characterization of metric invariance for fibered Riemannian manifolds.

Implications for Geometric Representation Learning

The principal conclusion of the present work is that local metric compatibility does not, by itself, imply globally consistent transport. An Ehresmann connection may induce horizontal distributions that preserve the Riemannian structure infinitesimally through the horizontal isometry determined by the submersion, while nevertheless exhibiting path dependence arising from nontrivial curvature and holonomy.

This distinction has a natural interpretation in modern representation learning, where highdimensional embedding spaces are often modeled as geometric objects equipped with locally defined similarity measures. Neighborhood-based constructions may provide local metric consistency, ensuring that nearby representations preserve semantic relationships under infinitesimal transport. However, when information is propagated along different trajectories through the representation space, accumulated transformations need not coincide if the underlying connection possesses nonvanishing curvature.

From this perspective, the holonomy of the connection quantifies the global obstruction to path-independent transport. Even when local neighborhoods admit compatible metric structures, transporting a representation around distinct loops may produce different identifications upon returning to the same point. Thus, curvature governs the failure of local compatibility to extend to a globally coherent transport mechanism.

Consequently, the framework developed in this paper suggests that the geometric analysis of learned representation spaces should distinguish between local preservation of metric structure and global consistency of transport. The interaction between Ehresmann connections, curvature, and holonomy provides a natural differential-geometric language for formalizing this distinction and offers a conceptual bridge between the theory of Riemannian submersions and emerging geometric approaches to representation learning.

Limitations

Several limitations of the present analysis should be emphasized. First, the framework assumes smooth manifolds and smooth Ehresmann connections throughout. Singular fibrations, orbifold structures, or spaces with lower regularity are not considered.

Second, metric-preserving transport and Riemannian submersion compatibility are imposed as structural hypotheses rather than derived from more primitive geometric conditions. Determining natural classes of connections for which these assumptions arise intrinsically remains an open problem.

Third, the global characterization relies on flatness of the horizontal connection. In many geometrically interesting settings, including nontrivial principal bundles and curved submersions, curvature and holonomy are expected to be nonvanishing, requiring weaker notions of approximate or local consistency.

Finally, the presentation focuses exclusively on deterministic smooth geometry and does not address stochastic transport, singular foliations, or discrete analogues.

Future Directions

The results suggest several natural directions for further study. One direction is the extension of the present framework to general Ehresmann connections with nonvanishing curvature, seeking quantitative relationships between curvature invariants, holonomy groups, and metric distortion.

A second direction is the investigation of probabilistic or metric measure analogues in which horizontal transport is replaced by random or variational processes while retaining geometric compatibility. Another promising avenue is the interaction with broader aspects of Riemannian submersion theory, including curvature identities, comparison theorems, and rigidity phenomena associated with O'Neill's fundamental tensors.

Finally, it would be of interest to determine whether analogous connection–curvature–holonomy principles extend to more general geometric settings, such as foliated manifolds, Lie groupoids, or higher-categorical structures.

Concluding Perspective

The viewpoint developed here suggests that metric compatibility and holonomy should be regarded as complementary aspects of a single geometric mechanism. Metric-preserving transport governs local isometric behavior, while curvature controls the emergence of global obstructions through holonomy. In this sense, the preservation of geometry under horizontal transport is characterized not solely by the choice of connection or metric, but by the interplay between connection, curvature, and Riemannian submersion structure.

Appendix: Geometric Interpretation for Representation-Based Systems

The geometric framework developed in the main text is formulated entirely within the language of differential geometry. Nevertheless, it admits a natural interpretation in settings where points of a manifold encode internal states and a projection identifies equivalence classes of such states.
In this perspective, the data

$$(M, Z, \pi, H, g_M, g_Z)$$

may be viewed as a geometric model of a representation system. The total space M describes a space of fine-grained configurations, the base manifold Z records their quotient under the projection

$$\pi : M \to Z,$$

and the horizontal distribution H specifies admissible transport between fibers. The Riemannian metrics gM and gZ endow these spaces with intrinsic notions of distance and local geometry.

The following observations place the results of the paper into this broader conceptual context without introducing additional mathematical assumptions.

O'Neill Interpretation

For a Riemannian submersion

$$\pi : (M, g_M) \to (Z, g_Z),$$

the horizontal distribution determines how geometric information in the total space projects onto the base manifold.
The O'Neill tensor

$$A_X Y = (\nabla_X Y)^V,$$

measures the vertical component generated by differentiating horizontal vector fields. Consequently, it quantifies the failure of the horizontal distribution to be integrable and characterizes the coupling between the horizontal and vertical geometries.

When the restriction

$$d\pi|_{H_x} : H_x \to T_{\pi(x)}Z$$

is an isometry, horizontal lengths and inner products are faithfully identified with their counterparts on the base manifold. In this sense, the Riemannian submersion transfers horizontal geometric structure from M to Z.

Ambrose–Singer Interpretation

The curvature of the Ehresmann connection measures the infinitesimal failure of horizontal transport to be path independent. Through the Ambrose–Singer theorem, this curvature generates the associated holonomy algebra and therefore determines the local infinitesimal source of holonomy.

From this viewpoint, holonomy records the cumulative effect of curvature encountered along closed horizontal loops. Even when local transport is metric preserving, nonvanishing curvature may induce a nontrivial global transformation after traversing a loop, reflecting the intrinsic geometry of the bundle.

Conceptual Consequence

Taken together, the results of the present work admit the following interpretation.

- Metric-preserving transport provides local geometric consistency by preserving inner products and lengths within the horizontal distribution.
- Riemannian submersion compatibility transfers this local consistency to the base manifold by identifying horizontal geometry with the induced geometry on Z.
- Holonomy constitutes the global obstruction to path-independent identification between fibers, with its infinitesimal structure governed by the curvature of the underlying connection.

Accordingly, in representation-theoretic applications where points of M encode internal states and points of Z encode observable or quotient-level information, curvature-induced holonomy provides a natural geometric mechanism for history-dependent evolution: identical endpoints in the base may correspond to distinct transported states whenever horizontal transport accumulates nontrivial holonomy.

This interpretation is intended solely as a conceptual application of the differential-geometric framework developed in the main text. The mathematical results themselves depend only on the properties of Ehresmann connections, Riemannian submersions, and the relationship between curvature and holonomy.

Highlights

- We study Ehresmann connections on smooth fiber bundles equipped with Riemannian metrics on both total space and base.
- We characterize metric-preserving horizontal transport within the framework of Riemannian submersions.
- We relate curvature of the horizontal distribution to O'Neill tensors and their effect on metric compatibility.
- We identify holonomy as the global obstruction to path-independent horizontal transport via the Ambrose–Singer correspondence.
- We provide a unified global geometric description separating local metric compatibility from global topological obstructions.

Declarations

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Data Statement

This work is purely theoretical in nature and develops results in global differential geometry. No empirical data were generated or analyzed. All results are derived analytically within the framework of Ehresmann connections, Riemannian submersions, curvature theory, and holonomy.

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