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Deformations and their Controlling Cohomologies of Nonabelian Embedding Tensors ON 3-Lie Algebras

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Abstract

In this paper, first we introduce the notion of a nonabelian embedding tensor on a 3-Lie algebra. In accordance with the general principles of deformation theories, a deformation theory of nonabelian embedding tensors is established. On the one hand, using the higher derived brackets, we construct an L_∞ -algebra whose Maurer-Cartan elements are nonabelian embedding tensors. Consequently, given a nonabelian embedding tensor T on a 3-Lie algebras, we obtain the twisted L_∞ -algebra that controls deformations of T . On the other hand, a 3-Lie algebra with a coherent action is identified from a nonabelian embedding tensor T such that the corresponding Loday-Pirashvili cohomology controls deformations of T . As applications, we use the second cohomology group to study infinitesimal deformations of nonabelian embedding tensors. In particular, we introduce the notion of a Nijenhuis elements on a 3-Lie algebras to characterize trivial infinitesimal deformations.

Mathematics Subject Classification.

Keywords: Nonabelian Embedding Tensors, Maurer-Cartan Characterization, Cohomologies, Deformations

Introduction

The notion of ternary Lie algebras appeared first in Nambu generalization of Hamiltonian mechanics which use a generalization of Poisson algebras with a ternary bracket [1]. The algebraic formulation of Nambu mechanics due to Takhtajan while the structure of n -Lie algebra was studied by Filippov, then completed by Kasmov in [2,3]. The structure of 3-Lie algebras play an important role in the supersymmetry and gauge symmetry transformations of the worldvolume theory of multiple coincident $M2$ -branes and is applied to the Bagger-Lambert theory has a novel local gauge symmetry which is based on the metric 3-Lie algebras (see) Casas, Loday and Pirashvili introduced the n -ary version of Leibniz algebras, which is called n -Leibniz algebras, whose skew-symmetric counterparts are n -Lie algebras [4-10]. In, the author introduced a graded Lie brackets on the space of cochains of n -Leibniz algebras and describe an n -Leibniz algebra structure as a canonical structure [11].

The concept of embedding tensors have been extensively studied and applied in many various areas of mathematics and physics. Embedding tensors first appeared in the study of gauged supergravity theory [12]. Using embedding tensors the $N = 8$ supersymmetric gauge theories and the Bagger-Lambert theory of multiple $M2$ -branes were both examined in [13]. Also Kotov and Strobl used its to construct tensor hierarchies, which prove the possible mathematical properties of embedding tensors from the physics point of view [14]. In mathematics, embedding tensors are said to be averaging operators. Averaging operators on arbitrary types of algebras such as associative algebras and Lie algebras were studied in [15,16]. In particular, an averaging operator on a Lie algebra can leads to a Leibniz algebra structure. In addition, averaging operators are closely associated to the operad theory and double algebras theory. For more details see [17-19].

Schlessinger and Stashe introduced L_∞ -algebras, and it has been used extensively in mathematical physics [20,21]. A Lie k -algebra is a concept that was first introduced and studied in [22]. For many applications, see [23,24]. Only the k -ary bracket is nontrivial in a Lie k -algebra, a unique type of L_∞ -algebra. Lie 3-algebras and 3-Lie algebras are both

ternary bracket algebras, but they are not the same. The controlling algebras of embedding tensors were given in [19].

The research on the deformation theory of algebraic structures starting with the seminal work of Gerstenhaber for associative algebras [25]. Next, Nijenhuis and Richardson expanded the study of deformation theory to Lie algebra [26,27]. The deformation problem of 3-Lie algebras and n-Lie algebras were studied respectively in and [26,28,29]. for more details see [30,31]. A good tool for connecting invariants to a mathematical structure is cohomology. It is established, in particular, that deformations are governed by proper cohomologies. A number of various algebraic structures are associative algebras, Leibniz algebras, Lie algebras, pre-Lie algebras, and n-Lie algebras. Recently, the deformaions and cohomologies theories for embedding tensor on Lie algebras is considered in [19,32,33].

Main Results and Outline of the Paper. The purpose of this paper is to study nonabelian embedding tensors on 3-Lie algebras and associated structures. For this purpose, first we study actions of a 3-Lie algebra A on a 3-Lie algebra \mathfrak{h} , which is totally different from the case of 3-Lie algebra actions. next, we introduce the notion of a nonabelian embedding tensor from a 3-Lie algebra \mathfrak{h} to a 3-Lie algebra A with respect to a coherent action $(\mathfrak{h}; \rho)$, and characterize it using graphs of the hemisemidirect product 3-Leibniz algebra. A nonabelian embedding tensor on 3-Lie algebra A with respect to a coherent action $\rho : \Lambda^2 A \rightarrow Der(\mathfrak{h})$ is a linear map $T : \mathfrak{h} \rightarrow A$ satisfying

$$[Tu, Tv, Tw]_A = T(\rho(Tu, Tv)w + [u, v, w]_{\mathfrak{h}}), \quad \forall u, v, w \in \mathfrak{h}.$$

Then, we use Voronovs higher derived brackets to construct an L_{∞} -algebra whose Maurer-Cartan elements are nonabelian embedding tensors. Consequently, we obtain the L_{∞} -algebra, it is a differential graded Lie algebra governing deformations of a nonabelian embedding tensor. Furthermore, a nonabelian embedding tensor leads to a new coherent action and the corresponding cohomology of 3-Leibniz algebras is taken to be the cohomology of the nonabelian embedding tensors. As applications, we use the second cohomology group to study infinitesimal deformations of a nonabelian embedding tensors T on a 3-Lie algebra using the previous cohomology theory. In particular, we introduce the notion of a Nijenhuis element associated to a nonabelian embedding tensors T on a 3-Lie algebra, which leads to a trivial infinitesimal deformation of the nonabelian embedding tensors T on 3-Lie algebra. Also we study their relationship with the infinitesimal deformations of the associated descendent 3-Leibniz algebra.

The Paper is Organized as Follows. In Section 2, we introduce the notion of a nonabelian embedding tensor on 3-Lie algebra A with respect to a coherent action (\mathfrak{h}, ρ) and illustrate the relationship between nonabelian embedding tensors and 3-Leibniz algebras. In Section 3, we construct an L_{∞} -algebra whose Maurer-Cartan elements are precisely nonabelian embedding tensor on 3-Lie algebras. We also using Getzlers twisted L_{∞} -algebra theory to characterize deformations of nonabelian embedding tensors on 3-Lie algebras. In Section 4, we introduce the cohomology theory of nonabelian embedding tensors on 3-Lie algebras. In Section 5, we study infinitesimal deformations of a nonabelian embedding tensors T on a 3-Lie algebra using the cohomology theory given in section 4.

In this paper, we work over an algebraically closed field \mathbb{K} of characteristic 0 and all the vector spaces are over \mathbb{K} finite dimensional.

2. Nonabelian Embedding Tensors and 3-Leibniz Algebras

In this section, first we introduce the notion of nonabelian embedding tensors on a 3-Lie algebras, which can be characterized by the graphs of the hemisemidirect product 3-Leibniz algebras.

Definition 2.1. [35] A **3-Lie algebra** is a vector space A equipped with a skew-symmetric trilinear map $[\cdot, \cdot, \cdot]_A : A \wedge A \wedge A \rightarrow A$, such that for $x_i \in A$, $1 \leq i \leq 5$, the following **fundamental identity** holds:

$$\begin{aligned} & [x_1, x_2, [x_3, x_4, x_5]_A]_A \\ &= [[x_1, x_2, x_3]_A, x_4, x_5]_A + [x_3, [x_1, x_2, x_4]_A, x_5]_A + [x_3, x_4, [x_1, x_2, x_5]_A]_A. \end{aligned} \tag{1}$$

Then ad_{x_1, x_2} is a derivation, i.e,

$$\text{ad}_{x_1, x_2}[x_3, x_4, x_5]_A = [\text{ad}_{x_1, x_2}x_3, x_4, x_5]_A + [x_3, \text{ad}_{x_1, x_2}x_4, x_5]_A + [x_3, x_4, \text{ad}_{x_1, x_2}x_5]_A.$$

Definition 2.2. [29] A **representation** of 3-Lie algebra $(A, [\cdot, \cdot, \cdot]_A)$ is a pair (V, ρ) , where V is a vector space and $\rho : A \wedge A \rightarrow gl(V)$ is a linear map such that for all $x_1, x_2, x_3, x_4 \in A$,

$$\rho(x_1, x_2)\rho(x_3, x_4) = \rho([x_1, x_2, x_3]_A, x_4) + \rho(x_3, [x_1, x_2, x_4]_A) + \rho(x_3, x_4)\rho(x_1, x_2), \tag{2}$$

$$\rho([x_1, x_2, x_3]_A, x_4) = \rho(x_1, x_2)\rho(x_3, x_4) + \rho(x_2, x_3)\rho(x_1, x_4) + \rho(x_3, x_1)\rho(x_2, x_4). \tag{3}$$

Let $(A, [\cdot, \cdot, \cdot]_A)$ and $(\mathfrak{h}, [\cdot, \cdot, \cdot]_{\mathfrak{h}})$ be two 3-Lie algebras. A 3-Lie algebras homomorphism $\rho : A \wedge A \rightarrow Der(\mathfrak{h})$ is said to be an **action** of A on \mathfrak{h} . An action ρ of A on \mathfrak{h} is called a **coherent action** if it satisfies

$$[\rho(x_1, x_2)u, v, w]_{\mathfrak{h}} = 0, \quad \forall x_1, x_2 \in A \text{ and } u, v, w \in \mathfrak{h}. \quad (4)$$

We denote by $(\mathfrak{h}; \rho)$ the coherent action. Denote by $\overline{Der(\mathfrak{h})} \subset Der(\mathfrak{h})$ the 3-Lie subalgebra of the derivation 3-Lie algebra $Der(\mathfrak{h})$ satisfying

$$\overline{Der(\mathfrak{h})} = \{D \in Der(\mathfrak{h}) \mid [Du, v, w]_{\mathfrak{h}} = 0, \quad \forall u, v, w \in \mathfrak{h}\}.$$

The 3-Lie algebra $\overline{Der(\mathfrak{h})}$ is said to be the **coherent derivation 3-Lie algebra**. A coherent action can be alternatively described by an homomorphism from the 3-Lie algebra A to the coherent derivation 3-Lie algebra $\overline{Der(\mathfrak{h})}$

Example 2.1. The adjoint action $ad : \wedge^2 A \rightarrow gl(A)$ is a coherent action of a **two-step nilpotent 3-Lie algebra** (see [19]) on itself and we have $\overline{Der(\mathfrak{h})} = Der(A)$.

Definition 2.3. A linear map $T : \mathfrak{h} \rightarrow A$ is called a **nonabelian embedding tensor** on 3-Lie algebra $(A, [\cdot, \cdot, \cdot]_A)$ with respect to a coherent action $(\mathfrak{h}; \rho)$ if T satisfies

$$[Tu, Tv, Tw]_A = T(\rho(Tu, Tv)w + [u, v, w]_{\mathfrak{h}}), \quad \forall u, v, w \in \mathfrak{h}. \quad (5)$$

Definition 2.4. Let T and T' be two nonabelian embedding tensors on 3-Lie algebra A with respect to a coherent action $(\mathfrak{h}; \rho)$. A **homomorphism** from T' to T consists of two 3-Lie algebras homomorphisms $\psi_A : A \rightarrow A$ and $\psi_{\mathfrak{h}} : \mathfrak{h} \rightarrow \mathfrak{h}$ such that

$$T \circ \psi_{\mathfrak{h}} = \psi_A \circ T', \quad (6)$$

$$\psi_{\mathfrak{h}}(\rho(x, y)u) = \rho(\psi_A(x), \psi_A(y))(\psi_{\mathfrak{h}}(u)), \quad \forall x, y \in A, u \in \mathfrak{h}. \quad (7)$$

In particular, if ψ_A and $\psi_{\mathfrak{h}}$ are invertible, $(\psi_A, \psi_{\mathfrak{h}})$ is said to be an isomorphism from T' to T . Consider the direct sum 3-Lie algebra $\overline{Der(\mathfrak{h})} \oplus \mathfrak{h}$, in which the 3-Lie bracket is defined by

$$[a + u, b + v, c + w] = [a, b, c] + [u, v, w]_{\mathfrak{h}}, \quad \forall a, b, c \in \overline{Der(\mathfrak{h})} \text{ and } u, v, w \in \mathfrak{h}.$$

Given $\rho : \overline{Der(\mathfrak{h})} \wedge \overline{Der(\mathfrak{h})} \rightarrow gl(\overline{Der(\mathfrak{h})} \oplus \mathfrak{h})$ by

$$\rho(a, b)(c + w) = (a + b)w, \quad \forall a, b, c \in \overline{Der(\mathfrak{h})} \text{ and } w \in \mathfrak{h}.$$

Then it is straightforward to deduce that ρ is a coherent action of the 3-Lie algebra $\overline{Der(\mathfrak{h})}$ on the 3-Lie algebra $\overline{Der(\mathfrak{h})} \oplus \mathfrak{h}$,

Proposition 2.2. The projection $pr : \overline{Der(\mathfrak{h})} \oplus \mathfrak{h} \rightarrow \overline{Der(\mathfrak{h})}$ is a nonabelian embedding tensor on the 3-Lie algebra $Der(\mathfrak{h})$ with respect to the coherent action $(Der(\mathfrak{h}) \oplus \mathfrak{h}; \rho)$.

Proof. It follows from

$$pr(\rho(pr(a + u), pr(b + v))(c + w) + [a + u, b + v, c + w]) = [a, b, c] = [pr(a + u), pr(b + v), pr(c + w)],$$

for all $a, b, c \in \overline{Der(\mathfrak{h})}$ and $u, v, w \in \mathfrak{h}$. \square

Example 2.3. There is a unique non-abelian 3-dimensional 3-Lie algebra. It has a basis $\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$ with respect to which the non-zero multiplication is defined by

$$[\varepsilon_1, \varepsilon_2, \varepsilon_3] = \varepsilon_1$$

Consider a linear transformation T given by $T = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ with respect to the basis $\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$ It

straightforward to check that T is a nonabelian embedding tensor with respect to the adjoint action if and only if

$$[T\varepsilon_1, T\varepsilon_2, T\varepsilon_3] = T([T\varepsilon_1, T\varepsilon_2, \varepsilon_3] + [\varepsilon_1, \varepsilon_2, \varepsilon_3]). \quad (8)$$

We have

$$\begin{aligned} & [T\varepsilon_1, T\varepsilon_2, T\varepsilon_3] \\ &= [a_{11}\varepsilon_1, a_{22}\varepsilon_2, a_{33}\varepsilon_3] + [a_{11}\varepsilon_1, a_{32}\varepsilon_3, a_{23}\varepsilon_2] + [a_{21}\varepsilon_2, a_{12}\varepsilon_1, a_{33}\varepsilon_3] \\ &+ [a_{21}\varepsilon_2, a_{32}\varepsilon_3, a_{13}\varepsilon_1] + [a_{31}\varepsilon_3, a_{12}\varepsilon_1, a_{23}\varepsilon_2] + [a_{31}\varepsilon_3, a_{22}\varepsilon_2, a_{13}\varepsilon_1] \end{aligned}$$

$$= (a_{11}a_{22}a_{33} - a_{11}a_{32}a_{23} - a_{21}a_{12}a_{33} + a_{21}a_{32}a_{13} + a_{31}a_{12}a_{23} - a_{31}a_{22}a_{13})\varepsilon_1$$

and

$$\begin{aligned} & T([T\varepsilon_1, T\varepsilon_2, \varepsilon_3] + [\varepsilon_1, \varepsilon_2, \varepsilon_3]) \\ &= T[a_{11}\varepsilon_1, a_{22}\varepsilon_2, \varepsilon_3] - T[a_{12}\varepsilon_1, a_{21}\varepsilon_2, \varepsilon_3] + T[\varepsilon_1, \varepsilon_2, \varepsilon_3] \\ &= (a_{11}a_{22} - a_{12}a_{21} + 1)a_{11}\varepsilon_1 \\ &+ (a_{11}a_{22} - a_{12}a_{21} + 1)a_{21}\varepsilon_2 \\ &+ (a_{11}a_{22} - a_{12}a_{21} + 1)a_{31}\varepsilon_3 \end{aligned}$$

Thus, T is a nonabelian embedding tensor if and only if

$$\begin{aligned} & a_{11}a_{22}a_{33} - a_{11}a_{32}a_{23} - a_{21}a_{12}a_{33} + a_{21}a_{32}a_{13} + a_{31}a_{12}a_{23} - a_{31}a_{22}a_{13} \\ &= (a_{11}a_{22} - a_{12}a_{21} + 1)a_{11} \end{aligned}$$

and

$$\begin{aligned} & (a_{11}a_{22} - a_{12}a_{21} + 1)a_{21} \\ &= (a_{11}a_{22} - a_{12}a_{21} + 1)a_{31} \\ &= 0. \end{aligned}$$

Summarize the above discussion, in particular, $T = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{11} \end{pmatrix}$ are nonabelian embedding tensors with respect to the adjoint action.

The algebraic structure underlying a nonabelian embedding tensor on the 3-Lie algebra $(A, [\cdot, \cdot, \cdot]_A)$ is a 3-Leibniz algebra.

Definition 2.5. [16] A **3-Leibniz algebra** is a vector space \mathcal{L} together with a trilinear map $[\cdot, \cdot, \cdot]_{\mathcal{L}} : \mathcal{L} \wedge \mathcal{L} \wedge \mathcal{L} \rightarrow \mathcal{L}$ such that, for all $x, y, u, v, w \in \mathcal{L}$

$$[x, y, [u, v, w]_{\mathcal{L}}]_{\mathcal{L}} = [[x, y, u]_{\mathcal{L}}, v, w]_{\mathcal{L}} + [u, [x, y, v]_{\mathcal{L}}, w]_{\mathcal{L}} + [u, v, [x, y, w]_{\mathcal{L}}]_{\mathcal{L}}. \quad (9)$$

Definition 2.6. A **representation** of a 3-Leibniz algebras $(\mathcal{L}, [\cdot, \cdot, \cdot]_{\mathcal{L}})$ is a quadruple (V, l, m, r) , where V is a vector space, $l, m, r : \mathcal{L} \wedge \mathcal{L} \rightarrow gl(V)$ are linear maps, such that the following equalities hold:

$$l(x, y)l(u, v) = l([x, y, u]_{\mathcal{L}}, v) + l(u, [x, y, v]_{\mathcal{L}}) + l(u, v)l(x, y), \quad (10)$$

$$l(x, y)l(u, v) = m([x, y, u]_{\mathcal{L}}, v) + m(u, [x, y, v]_{\mathcal{L}}) + m(u, v)l(x, y), \quad (11)$$

$$l(x, y)l(u, v) = r([x, y, u]_{\mathcal{L}}, v) + r(u, [x, y, v]_{\mathcal{L}}) + r(u, v)l(x, y), \quad (12)$$

$$m(x, [y, u, v]_{\mathcal{L}}) = r(u, v)m(x, y) + m(y, v)m(x, u) + l(y, u)m(x, v), \quad (13)$$

$$r(x, [y, u, v]_{\mathcal{L}}) = r(u, v)m(x, y) + m(y, v)r(x, u) + l(y, u)r(x, v), \quad (14)$$

for all $x, y, u, v \in \mathcal{L}$.

Proposition 2.4. Let $(A, [\cdot, \cdot, \cdot]_A)$ and $(\mathfrak{h}, [\cdot, \cdot, \cdot]_{\mathfrak{h}})$ be 3-Lie algebras. Let $\rho : A \wedge A \rightarrow Der(\mathfrak{h})$ be a coherent action of A on \mathfrak{h} . Then, the direct sum $A \oplus \mathfrak{h}$ carries a 3-Leibniz algebra structure given by

$$[x + u, y + v, z + w]_{\rho} = [x, y, z]_A + \rho(x, y)w + [u, v, w]_{\mathfrak{h}}, \quad \forall x, y, z \in A \text{ and } u, v, w \in \mathfrak{h}, \quad (15)$$

which is called the **nonabelian hemisemidirect product**, denoted by $A \ltimes_{\rho}^{\mathfrak{h}} \mathfrak{h}$.

Proof. By the Jacobi identities for the 3-Lie brackets $[\cdot, \cdot, \cdot]_A$ and $[\cdot, \cdot, \cdot]_{\mathfrak{h}}$, and the fact that ρ is a homomorphism from the Lie algebra $(A, [\cdot, \cdot, \cdot]_A)$ to the derivation Lie algebra $Der(\mathfrak{h})$, we have

$$\begin{aligned} & [x_1 + u_1, x_2 + u_2, [x_3 + u_3, x_4 + u_4, x_5 + u_5]_{\rho}]_{\rho} - [[x_1 + u_1, x_2 + u_2, x_3 + u_3]_{\rho}, x_4 + u_4, x_5 + u_5]_{\rho} \\ & - [x_3 + u_3, [x_1 + u_1, x_2 + u_2, x_4 + u_4]_{\rho}, x_5 + u_5]_{\rho} - [x_3 + u_3, x_4 + u_4, [x_1 + u_1, x_2 + u_2, x_5 + u_5]_{\rho}]_{\rho} \\ & = [x_1, x_2, [x_3, x_4, x_5]_A]_A - [[x_1, x_2, x_3]_A, x_4, x_5]_A - [x_3, [x_1, x_2, x_4]_A, x_5]_A - [x_3, x_4, [x_1, x_2, x_5]_A]_A \\ & + [u_1, u_2, [u_3, u_4, u_5]_{\mathfrak{h}}]_{\mathfrak{h}} - [[u_1, u_2, u_3]_{\mathfrak{h}}, u_4, u_5]_{\mathfrak{h}} - [u_3, [u_1, u_2, u_4]_{\mathfrak{h}}, u_5]_{\mathfrak{h}} - [u_3, u_4, [u_1, u_2, u_5]_{\mathfrak{h}}]_{\mathfrak{h}} \\ & + \rho(x_1, x_2)[u_3, u_4, u_5]_{\mathfrak{h}} - \rho(x_1, x_2)u_3, u_4, u_5]_{\mathfrak{h}} - [u_3, \rho(x_1, x_2)u_4, u_5]_{\mathfrak{h}} - [u_3, u_4, \rho(x_1, x_2)u_5]_{\mathfrak{h}} \\ & + \rho(x_1, x_2)\rho(x_3, x_4)u_5 - \rho([x_1, x_2, x_3]_A, x_4)u_5 - \rho(x_3, [x_1, x_2, x_4]_A)u_5 - \rho(x_3, x_4)\rho(x_1, x_2)u_5 \end{aligned}$$

$$\begin{aligned}
& +[u_1, u_2, \rho(x_3, x_4)u_5]_{\mathfrak{h}} - [u_3, \rho(x_1, x_2)u_4, u_5]_{\mathfrak{h}} - \rho(x_3, x_4)[u_1, u_2, u_5]_{\mathfrak{h}} \\
& = [u_1, u_2, \rho(x_3, x_4)u_5]_{\mathfrak{h}} - [u_3, \rho(x_1, x_2)u_4, u_5]_{\mathfrak{h}} - \rho(x_3, x_4)[u_1, u_2, u_5]_{\mathfrak{h}}
\end{aligned}$$

Using the identity (15) of hemisemidirect product, one can characterize T by their graphs.

Proposition 2.5. *Let $(A, [\cdot, \cdot, \cdot]_A)$ and $(\mathfrak{h}, [\cdot, \cdot, \cdot]_{\mathfrak{h}})$ be 3-Lie algebras. Let $\rho : \wedge^2 A \rightarrow \text{Der}(\mathfrak{h})$ be a coherent action of A on \mathfrak{h} . A linear map $T : \mathfrak{h} \rightarrow A$ is a nonabelian embedding tensor on a 3-Lie algebra A with respect to the coherent action $(\mathfrak{h}; \rho)$ if and only if its **graph** $Gr(T) = \{Tu + u \mid u \in \mathfrak{h}\}$ is a 3-Leibniz subalgebra of the 3-Leibniz algebra $(A \oplus \mathfrak{h}, [\cdot, \cdot, \cdot]_{\rho})$.*

Proof. Let $Tu + u, Tv + v$ and $Tw + w \in Gr(T)$. Then we have

$$\begin{aligned}
& [Tu + u, Tv + v, Tw + w]_{\rho} \\
& = [Tu, Tv, Tw]_A + \rho(Tu, Tv)w + [u, v, w]_{\mathfrak{h}}.
\end{aligned}$$

Assume that $Gr(T)$ is a Leibniz subalgebra if and only if (5) holds, i.e. $T : \mathfrak{h} \rightarrow A$ is a nonabelian embedding tensor. Since \mathfrak{h} and \overline{Gr}_T are naturally isomorphic, we get the following conclusion immediately.

Since \overline{Gr}_T is isomorphic to \mathfrak{h} as a vector space by the identification $T(u) + u \cong u$. A nonabelian embedding tensor T induces a 3-Leibniz algebra structure on \mathfrak{h} given by

$$[u, v, w]_T = \rho(Tu, Tv)w + [u, v, w]_{\mathfrak{h}}, \quad \forall u, v, w \in \mathfrak{h}. \quad (16)$$

Moreover, T is a 3-Leibniz algebra homomorphism from the 3-Leibniz algebra $(\mathfrak{h}, [\cdot, \cdot, \cdot]_T)$ to the 3-Lie algebra $(A, [\cdot, \cdot, \cdot]_A)$.

The 3-Leibniz algebra $(\mathfrak{h}, [\cdot, \cdot, \cdot]_T)$ is called the **descendent 3-Leibniz algebra** of the nonabelian embedding tensor T .

Example 2.6. *The projection $pr : \overline{Der(\mathfrak{h})} \oplus \mathfrak{h} \rightarrow \overline{Der(\mathfrak{h})}$ is a nonabelian embedding tensor on the 3-Lie algebra $Der(\mathfrak{h})$ with respect to the coherent action $(\overline{Der(\mathfrak{h})} \oplus \mathfrak{h}; \rho)$, where*

$$\begin{aligned}
[a + u, b + v, c + w]_{pr} & = [a + u, b + v, c + w] + \rho(pr(a + u), pr(b + v))(c + w) \\
& = [a, b, c] + (a + b)w + [u, v, w]_{\mathfrak{h}},
\end{aligned}$$

for all $a; b; c \in \overline{Der(\mathfrak{h})}$ and $u; v; w \in \mathfrak{h}$.

Proposition 2.7. *Let T and T' be two nonabelian embedding tensors on a 3-Lie algebra A with respect to a coherent action $(\mathfrak{h}; \rho)$ and $(\psi_A, \psi_{\mathfrak{h}})$ a homomorphism from T' to T . Then $\psi_{\mathfrak{h}}$ is a homomorphism of 3-Leibniz algebras from $(\mathfrak{h}, [\cdot, \cdot, \cdot]_{T'})$ to $(\mathfrak{h}, [\cdot, \cdot, \cdot]_T)$.*

Proof. For all $u; v; w \in \mathfrak{h}$, we have

$$\begin{aligned}
& \psi_{\mathfrak{h}}([u, v, w]_{T'}) \\
& = \psi_{\mathfrak{h}}(\rho(T'u, T'v)w + [u, v, w]_{\mathfrak{h}}) = \rho(\psi_A(T'u), \psi_A(T'v))\psi_{\mathfrak{h}}(w) + [\psi_{\mathfrak{h}}(u), \psi_{\mathfrak{h}}(v), \psi_{\mathfrak{h}}(w)]_{\mathfrak{h}} \\
& = \rho(T\psi_{\mathfrak{h}}(u), T\psi_{\mathfrak{h}}(v))\psi_{\mathfrak{h}}(w) + [\psi_{\mathfrak{h}}(u), \psi_{\mathfrak{h}}(v), \psi_{\mathfrak{h}}(w)]_{\mathfrak{h}} = [\psi_{\mathfrak{h}}(u), \psi_{\mathfrak{h}}(v), \psi_{\mathfrak{h}}(w)]_T.
\end{aligned}$$

Then $\psi_{\mathfrak{h}}$ is a homomorphism of 3-Leibniz algebras from $(\mathfrak{h}, [\cdot, \cdot, \cdot]_{T'})$ to $(\mathfrak{h}, [\cdot, \cdot, \cdot]_T)$. □

3. Maurer-Cartan Elements and Nonabelian Embedding Tensors

In this section, given an action of 3-Lie algebras, we construct an L_{∞} -algebra whose Maurer-Cartan elements are nonabelian embedding tensors T on a 3-Lie algebra. Then we obtain the L_{∞} -algebra that controls deformations of nonabelian embedding tensors T on 3-Lie algebras. This fact can be observed as certain justification of nonabelian embedding tensors T on a 3-Lie algebra being interesting structures.

Definition 3.1. [30] A **Maurer-Cartan element** of an L_{∞} -algebra $(A = \bigoplus_{k \in \mathbb{Z}^+} A^k, \{l_i\}_{i=1}^{+\infty})$ is an element $\alpha \in A^0$ satisfying the Maurer-Cartan equation

$$\sum_{n=1}^{+\infty} \frac{1}{n!} l_n(\alpha, \dots, \alpha) = 0. \quad (17)$$

Let α be a Maurer-Cartan element of an L_∞ -algebra $(A, \{l_i\}_{i=1}^{+\infty})$. For all $k \geq 1$ and $x_1, \dots, x_n \in A$, define a series of linear maps $l_k^\alpha : \wedge^k A \rightarrow A$ of degree 1 by

$$l_k^\alpha(x_1, \dots, x_k) = \sum_{n=0}^{+\infty} \frac{1}{n!} l_{n+k} \underbrace{\{\alpha, \dots, \alpha, x_1, \dots, x_n\}}_n. \quad (18)$$

Theorem 3.1. *With the above notations, $(A, \{l_i^\alpha\}_{i=1}^{+\infty})$ is an L_∞ -algebra, obtained from the L_∞ -algebra $(A, \{l_i\}_{i=1}^{+\infty})$ by twisting with the Maurer-Cartan element α . Moreover, $\alpha + \alpha'$ is a Maurer-Cartan element of $(A, [\cdot, \cdot]_A)$ if and only if θ is a Maurer-Cartan element of the twisted L_∞ -algebra $(A, l_1^\alpha, l_2^\alpha, l_3^\alpha)$.*

now, we recall that in [33], Th. Voronov extended the theory of higher derived brackets, which is a useful tool to construct explicit L_∞ -algebras.

Definition 3.2. [33] A *V-data* consists of a quadruple $(L, F, \mathcal{P}, \Delta)$, where

- * $(L, [-, -])$ is a graded Lie algebra,
- * F is an abelian graded Lie subalgebra of $(L, [-, -])$,
- * $\mathcal{P} : L \rightarrow L$ is a projection, that is $\mathcal{P} \circ \mathcal{P} = \mathcal{P}$, whose image is F and kernel is a graded Lie subalgebra of $(L, [-, -])$,
- * Δ is an element in $\ker(\mathcal{P})^1$ such that $[\Delta, \Delta] = 0$.

Theorem 3.2. [33] *Let $(L, F, \mathcal{P}, \Delta)$ be a V-data. Then $(F, \{l_k\}_{k=1}^{+\infty})$ is an L_∞ -algebra, where*

$$l_k(a_1, \dots, a_k) = \mathcal{P} \underbrace{[\dots]}_k [[\Delta, a_1], a_2], \dots, a_k], \text{ for homogeneous } a_1, \dots, a_k \in F. \quad (19)$$

We say $\{l_k\}_{k=1}^{+\infty}$ the **higher derived brackets** of the V-data $(L, F, \mathcal{P}, \Delta)$.

Let A be a vector space. We consider the graded vector space

$$C^*(A, A) = \bigoplus_{n \geq 0} C^n(A, A) = \bigoplus_{n \geq 0} \text{Hom}(\underbrace{\wedge^2 A \otimes \dots \otimes \wedge^2 A}_n \wedge A, A).$$

Theorem 3.3. [23] *The graded vector space $C^*(A; A)$ together with the graded bracket*

$$[P, Q] = P \circ Q - (-1)^{qp} Q \circ P, \quad \forall P \in C^p(A, A) \text{ and } Q \in C^q(A, A), \quad (20)$$

is a graded Lie algebra, where $P \circ Q \in C^{p+q}(A, A)$ is given as follows:

$$\begin{aligned} & (P \circ Q)(\mathfrak{X}_1, \dots, \mathfrak{X}_{q+p}, x) \\ &= \sum_{k=1}^q (-1)^{(k-1)p} \sum_{\sigma \in S(k-1, p)} \text{sgn}(\sigma) P(\mathfrak{X}_{\sigma(1)}, \dots, \mathfrak{X}_{\sigma(k-1)}, Q(\mathfrak{X}_{\sigma(k)}, \dots, \mathfrak{X}_{\sigma(k+p-1)}, x_{k+p})) \\ & \quad \wedge y_{k+p}, \mathfrak{X}_{k+p+1}, \dots, \mathfrak{X}_{q+p}, x) \\ &+ \sum_{k=1}^q (-1)^{(k-1)p} \sum_{\sigma \in S(k-1, p)} \text{sgn}(\sigma) P(\mathfrak{X}_{\sigma(1)}, \dots, \mathfrak{X}_{\sigma(k-1)}, x_{k+p}) \\ & \quad \wedge Q(\mathfrak{X}_{\sigma(k)}, \dots, \mathfrak{X}_{\sigma(k+p-1)}, y_{k+p}) \wedge y_{k+p}, \mathfrak{X}_{k+p+1}, \dots, \mathfrak{X}_{q+p}, x) \\ &+ \sum_{\sigma \in S(q, p)} (-1)^{qp} \text{sgn}(\sigma) P(\mathfrak{X}_{\sigma(1)}, \dots, \mathfrak{X}_{\sigma(q)}, Q(\mathfrak{X}_{\sigma(q+1)}, \dots, \mathfrak{X}_{\sigma(q+p)}, x)). \end{aligned}$$

for all $\mathfrak{X}_i = x_i \wedge y_i \in \wedge^2 A$, $i = 1, 2, \dots, q + p$ and $x \in A$.

Remark 3.1. Let $\Omega \in C^2(A, A)$, so we have

$$\begin{aligned} \frac{1}{2} [\Omega, \Omega]_{3Leib} &= \Omega \circ \Omega(\mathfrak{X}_1, \mathfrak{X}_2, x) \\ &= \Omega(\Omega(\mathfrak{X}_1, y_1) \wedge y_2, x) + \Omega(y_1 \wedge \Omega(\mathfrak{X}_1, y_2), x) - \Omega(\mathfrak{X}_1, \Omega(\mathfrak{X}_2, x)) + \Omega(\mathfrak{X}_2, \Omega(\mathfrak{X}_1, x)). \end{aligned}$$

where, $\mathfrak{X}_1 = x_1 \wedge x_2 \in \wedge^2 A$, $\mathfrak{X}_2 = y_1 \wedge y_2 \in \wedge^2 A$ and $x \in A$. Moreover, is a 3-Leibniz bracket if and only if $[\Omega, \Omega]_{3Leib} = 0$, i.e, Ω is a Maurer-Cartan element of $(C^*(A, A), [-, -]_{3Leib})$.

Let $\rho : \wedge^2 A \rightarrow Der(\mathfrak{h})$ be an action of a 3-Lie algebra $(A, [\cdot, \cdot, \cdot]_A)$ on a 3-Lie algebra $(\mathfrak{h}, [\cdot, \cdot, \cdot]_{\mathfrak{h}})$. For convenience, we use $\pi : \wedge^3 A \rightarrow A$ to indicate the 3-Lie bracket $[\cdot, \cdot, \cdot]_A$ and $\mu : \wedge^3 \mathfrak{h} \rightarrow \mathfrak{h}$ to indicate the 3-Lie bracket $[\cdot, \cdot, \cdot]_{\mathfrak{h}}$. In the sequel, we use $\pi \boxplus \rho \boxplus \mu$ to denote the element in $Hom(\wedge^3(A \oplus \mathfrak{h}), A \oplus \mathfrak{h})$ defined by

$$(\pi \boxplus \rho \boxplus \mu)(x + u, y + v, z + w) = [x, y, z]_A + \rho(x, y)w + [u, v, w]_{\mathfrak{h}}, \tag{21}$$

for all $x, y, z \in A$, $u, v, w \in \mathfrak{h}$. Note that the right hand side is exactly the semidirect product 3-Leibniz algebra structure given in Proposition (3.5). Therefore, by remark (3.1), we have

$$[\pi \boxplus \rho \boxplus \mu, \pi \boxplus \rho \boxplus \mu]_{3Leib} = 0.$$

Lemma 3.4. *Let $T : \mathfrak{h} \rightarrow A$ be a nonabelian embedding tensor on a 3-Lie algebra $(A, [\cdot, \cdot, \cdot]_A)$ with respect to the coherent action $(\mathfrak{h}; \rho)$. For all $u, v, w \in \mathfrak{h}$, we have*

$$\begin{aligned} & [[\pi \boxplus \rho \boxplus \mu, T]_{3Leib}, T]_{3Leib}(x + u, y + v, y + z) \\ &= 2([Tu, Tv, z]_A + [Tu, y, Tw]_A + [x, Tv, Tw]_A - T(\rho(Tu, y)w + \rho(x, Tv)w) + \rho(Tu, Tv)w). \end{aligned} \tag{22}$$

Proof. For all $x, y, z \in A$ and $u, v, w \in \mathfrak{h}$,

$$\begin{aligned} & [[\pi \boxplus \rho \boxplus \mu, T]_{3-Leib}, T]_{3-Leib}(x + u, y + v, z + w) \\ &= [\pi \boxplus \rho \boxplus \mu, T]_{3-Leib}(T(x + u), y + v, z + w) + [\pi \boxplus \rho \boxplus \mu, T]_{3-Leib}(x + u, T(y + v), z + w) \\ & \quad + [\pi \boxplus \rho \boxplus \mu, T]_{3-Leib}(x + u, y + v, T(z + w)) - T[\pi \boxplus \rho \boxplus \mu, T]_{3-Leib}(x + u, z + w) \\ &= (\pi \boxplus \rho \boxplus \mu)(T(x + u), T(y + v), z + w) + (\pi \boxplus \rho \boxplus \mu)(T(x + u), y + v, T(z + w)) \\ & \quad - T(\pi \boxplus \rho \boxplus \mu)(T(x + u), y + v, z + w) + (\pi \boxplus \rho \boxplus \mu)(T(x + u), T(y + v), z + w) \\ & \quad + (\pi \boxplus \rho \boxplus \mu)(x + u, T(y + v), T(z + w)) - T(\pi \boxplus \rho \boxplus \mu)(x + u, T(y + v), z + w) \\ & \quad + (\pi \boxplus \rho \boxplus \mu)(T(x + u), y + v, T(z + w)) + (\pi \boxplus \rho \boxplus \mu)(x + u, T(y + v), T(z + w)) \\ & \quad - T(\pi \boxplus \rho \boxplus \mu)(x + u, y + v, T(z + w)) - T(\pi \boxplus \rho \boxplus \mu)(T(x + u), y + v, z + w) \\ & \quad - T(\pi \boxplus \rho \boxplus \mu)(x + u, T(y + v), z + w) - T(\pi \boxplus \rho \boxplus \mu)(x + u, y + v, T(z + w)) \\ &= 2([Tu, Tv, z]_A + [Tu, y, Tw]_A + [x, Tv, Tw]_A - T(\rho(Tu, y)w + \rho(x, Tv)w) + \rho(Tu, Tv)w). \end{aligned}$$

Proposition 3.5. *Let $(\mathfrak{h}; \rho)$ be a coherent action of 3-Lie algebra $(A, [\cdot, \cdot, \cdot]_A)$. Then we have a \mathfrak{h} -data $(L, f, \mathcal{P}, \Delta)$ as follows:*

- * $(C^*(A \oplus \mathfrak{h}, A \oplus \mathfrak{h}), [-, -]_{3Leib})$ is the graded Lie algebra,
- * $F = C^*(A, A) = \bigoplus_{n \geq 0} C^n(A, A) = \bigoplus_{n \geq 0} Hom(\underbrace{\wedge^2 A \otimes \dots \otimes \wedge^2 A}_n \wedge A, A)$ is the abelian graded Lie subalgebra of $(C^*(A \oplus \mathfrak{h}, A \oplus \mathfrak{h}), [-, -]_{3Leib})$,
- * $\mathcal{P} : L \rightarrow \hat{L}$ is the projection onto the subspace F ,
- * $\Delta = \pi \boxplus \rho \boxplus \mu$.

Then, we have a 3-Lie algebra $(C^*(\mathfrak{h}, A), [[\cdot, \cdot, \cdot]])$, where the trilinear product

$$[[\cdot, \cdot, \cdot]] : C^p(\mathfrak{h}, A) \times C^q(\mathfrak{h}, A) \times C^r(\mathfrak{h}, A) \rightarrow C^{p+q+r}(\mathfrak{h}, A)$$

is given by

$$[[P, Q, R]] = [[[\pi \boxplus \rho \boxplus \mu, P]_{3Leib}, Q]_{3Leib}, R]_{3Leib},$$

For all $P \in C^p(\mathfrak{h}, A)$, $Q \in C^q(\mathfrak{h}, A)$ and $R \in C^r(\mathfrak{h}, A)$.

Proof. by Theorem (3.2), $(F, \{l_k\}_{k=1}^{+\infty})$ is an L_∞ -algebra, where l_k is defined by (19). For all $P \in C^p(\mathfrak{h}, A)$, $Q \in C^q(\mathfrak{h}, A)$ and $R \in C^r(\mathfrak{h}, A)$, by lemma (3.4), we obtain

$$\begin{aligned} & [\pi \boxplus \rho \boxplus \mu, P]_{3Leib} \in \text{Ker } \mathcal{P}, \\ & [[\pi \boxplus \rho \boxplus \mu, P]_{3Leib}, Q]_{3Leib} \in \text{Ker } \mathcal{P}. \end{aligned}$$

Similarly, we have that $lk = 0$ when $k \geq 4$. Thus, the graded vector space $C^*(\mathfrak{h}, A)$ is a 3-Lie algebra with the trilinear multiplication $\llbracket \cdot, \cdot, \cdot \rrbracket$ and the other maps are trivial.

Theorem 3.6. *Let $(\mathfrak{h}; \rho)$ be a coherent action of a 3-Lie algebra $(A, [\cdot, \cdot, \cdot]_A)$. Then Maurer- Cartan elements of the 3-Lie algebra $(C^*(\mathfrak{h}, A), \llbracket \cdot, \cdot, \cdot \rrbracket)$ are precisely nonabelian embedding tensor on the 3-Lie algebra $(A, [\cdot, \cdot, \cdot]_A)$ with respect to the coherent action $(\mathfrak{h}; \rho)$.*

Proof. It is straightforward to obtain that

$$\begin{aligned} & \mathcal{P}[\llbracket [\pi \boxplus \rho \boxplus \mu, T]_{3Leib}, T \rrbracket_{3Leib}, T \rrbracket_{3Leib}(u, v, z) \\ &= 6(\pi(Tu, Tv, Tw) - T(\rho(Tu, Tv)w)) \\ & \mathcal{P}[\pi \boxplus \rho \boxplus \mu, T]_{3Leib}(u, v, w) = -T\mu(u, v, w), \end{aligned}$$

Let T be a Maurer-Cartan element of the L_∞ -algebra $(C^*(\mathfrak{h}, A), l_1, l_3)$. Then we obtain

$$\begin{aligned} & \sum_{n=1}^{+\infty} \frac{1}{n!} l_n(T, \dots, T)(u, v, w) \\ &= \mathcal{P}[\pi \boxplus \rho \boxplus \mu, T]_{3Leib}(u, v, w) + \frac{1}{3!} \mathcal{P}[\llbracket [\pi \boxplus \rho \boxplus \mu, T]_{3Leib}, T \rrbracket_{3Leib}, T \rrbracket_{3Leib}(u, v, w) \\ &= \pi(Tu, Tv, Tw) - T(\rho(Tu, Tv)w) + \mu(u, v, w) \\ &= 0, \end{aligned}$$

which implies that T is nonabelian embedding tensor on the 3-Lie algebra $(A, [\cdot, \cdot, \cdot]_A)$ with respect to the coherent action $(\mathfrak{h}; \rho)$.

Proposition 3.7. *Let T be a nonabelian embedding tensor from a 3-Lie algebra \mathfrak{h} to a 3-Lie algebra A with respect to a coherent action $(\mathfrak{h}; \rho)$. Then $C^*(\mathfrak{h}; A)$ carries a twisted L_∞ -algebra structure as following:*

$$l_1^T(P) = l_1(P) + \frac{1}{2} \llbracket T, T, P \rrbracket, \tag{23}$$

$$l_2^T(P, Q) = l_3 \llbracket T, P, Q \rrbracket, \tag{24}$$

$$l_3^T(P, Q, R) = l_3 \llbracket P, Q, R \rrbracket, \tag{25}$$

$$l_k^T = 0, \quad k \geq 4, \tag{26}$$

where $P \in C^p(\mathfrak{h}; \rho)$, $Q \in C^q(\mathfrak{h}; \rho)$ and $R \in C^r(\mathfrak{h}; \rho)$.

Proof. Since T is a Maurer-Cartan element of the L_∞ -algebra $(C^*(\mathfrak{h}, A), l_1, l_3)$, by Theorem (3.2), we have the conclusion.

The above L_∞ -algebra controls deformations of nonabelian embedding tensors on 3-Lie algebras

Theorem 3.8. *Let $T : \mathfrak{h} \rightarrow A$ be a nonabelian embedding tensor from a 3-Leibniz algebra $(\mathfrak{h}, [\cdot, \cdot, \cdot]_{\mathfrak{h}})$ to a 3-Lie algebra $(A, [\cdot, \cdot, \cdot]_A)$ with respect to a coherent action $(\mathfrak{h}; \rho)$. Then for a linear map $T' : \mathfrak{h} \rightarrow A$, $T + T'$ is a nonabelian embedding tensor if and only if T' is a Maurer-Cartan element of the twisted L_∞ -algebra $(C^*(\mathfrak{h}, A), l_1^T, l_2^T, l_3^T)$, that is T' satisfies the Maurer-Cartan equation:*

$$l_1^T(T') + \frac{1}{2} l_2^T(T', T') + \frac{1}{3!} l_3^T(T', T', T') = 0. \tag{27}$$

Proof. By Theorem (3.6) $T + T'$ is a nonabelian embedding tensor if and only if

$$l_1^T \llbracket T + T' \rrbracket + \frac{1}{3!} \llbracket T + T', T + T', T + T' \rrbracket = 0.$$

Applying $l_1(T) + \frac{1}{3!} l_3 \llbracket T, T, T \rrbracket = 0$, the above condition is equivalent to

$$l_1^T(T') + \frac{1}{2} l_2^T \llbracket T, T, T' \rrbracket + \frac{1}{2} l_3 \llbracket T, T', T' \rrbracket + \frac{1}{6} l_3 \llbracket T', T', T' \rrbracket = 0,$$

That is, $l_1^T(T') + \frac{1}{2}l_2^T(T', T') + \frac{1}{3!}l_3^T\llbracket T', T', T' \rrbracket = 0$, which implies that T' is a Maurer-Cartan element of the twisted L_∞ -algebra $(C^*(\mathfrak{h}, A), l_1^T, l_2^T, l_3^T)$.

4. Cohomology of Nonabelian Embedding Tensors on 3-Lie algebras

In this section, we construct a coherent action of the 3-Leibniz algebra $(\mathfrak{h}, [\cdot, \cdot, \cdot]_T)$ on the vector space A from a nonabelian embedding tensor $T : \mathfrak{h} \rightarrow A$ and we define a cohomology theory for nonabelian embedding tensors via the **Loday-Pirashvili cohomology** of the descendent 3- Leibniz algebra.

Set $\mathfrak{C}_{3Leib}^n(\mathcal{L}; V) = \text{Hom}(\underbrace{\wedge^2 \mathcal{L} \otimes \cdots \otimes \wedge^2 \mathcal{L}}_{n-1} \wedge \mathcal{L}, V)$, $(n \geq 1)$ the space of n -cochains. Consider the differential

$\partial : \mathfrak{C}_{3Leib}^n(\mathcal{L}; V) \rightarrow \mathfrak{C}_{3Leib}^{n+1}(\mathcal{L}; V)$ given by

$$\begin{aligned} & (\partial f)(\mathfrak{x}_1, \dots, \mathfrak{x}_n, z) \\ &= \sum_{1 \leq j < k \leq n} (-1)^j f(\mathfrak{x}_1, \dots, \hat{\mathfrak{x}}_j, \dots, \mathfrak{x}_{k-1}, [x_j, y_j, x_k]_{\mathcal{L}} \wedge y_k, \mathfrak{x}_{k+1}, \dots, \mathfrak{x}_n, z) \\ &+ \sum_{1 \leq j < k \leq n} (-1)^j f(\mathfrak{x}_1, \dots, \hat{\mathfrak{x}}_j, \dots, \mathfrak{x}_{k-1}, x_k \wedge [x_j, y_j, x_k]_{\mathcal{L}}, \mathfrak{x}_{k+1}, \dots, \mathfrak{x}_n, z) \\ &+ \sum_{j=1}^n ((-1)^j f(\mathfrak{x}_1, \dots, \hat{\mathfrak{x}}_j, \dots, \mathfrak{x}_n, [\mathfrak{x}_j, z]_{\mathcal{L}}) + (-1)^{j+1} l(\mathfrak{x}_j) f(\mathfrak{x}_1, \dots, \hat{\mathfrak{x}}_j, \dots, \mathfrak{x}_n, z)) \\ &+ (-1)^{n+1} (m(x_n, z) f(\mathfrak{x}_1, \dots, \mathfrak{x}_{n-1}, y_n) + r(y_n, z) f(\mathfrak{x}_1, \dots, \mathfrak{x}_{n-1}, x_n)). \end{aligned}$$

Where $\mathfrak{x}_i = x_i \wedge y_i \in \wedge^2 \mathcal{L}$, for all $i = 1, 2, \dots, n$ and $z \in \mathcal{L}$. It was shown in [16,20] that $\partial \circ \partial = 0$. Thus $(\oplus_{n=1}^{+\infty} \mathfrak{C}_{3Leib}^n(\mathcal{L}; V), \partial)$ is a cochain complex.

Definition 4.1. The quotient space $H_{3Leib}^n(\mathcal{L}; V) = Z_{3Leib}^n(\mathcal{L}; V) / B_{3Leib}^n(\mathcal{L}; V)$, where

$$Z_{3Leib}^n(\mathcal{L}; V) = \{f \in \mathfrak{C}_{3Leib}^n(\mathcal{L}; V) \mid \partial f = 0\}$$

is the space of n -cocycles and

is the space of n -coboundaries, is called the cohomology group of the 3-Leibniz algebra $(\mathcal{L}, [\cdot, \cdot, \cdot]_{\mathcal{L}})$ with coefficients in V .

Since eq (16) defines a 3-Leibniz algebra structure on \mathfrak{h} , then we have the following proposition.

Proposition 4.1. Let T be a nonabelian embedding tensor on a 3-Lie algebra $(A, [\cdot, \cdot, \cdot]_A)$ with respect to a coherent action $(\mathfrak{h}; \rho)$. Define linear maps $l_T, m_T, r_T : \wedge^2 \mathfrak{h} \rightarrow \mathfrak{gl}(A)$ by

$$l_T(u, v)(x) = [Tu, Tv, x]_A, \tag{28}$$

$$m_T(u, v)(x) = [Tu, x, Tv]_A - T(\rho(Tu, x)v), \tag{29}$$

$$r_T = -m_T, \tag{30}$$

for all $u, v \in \mathfrak{h}, x \in A$. Then $(A; l_T, m_T, r_T)$ is a representation of the 3-Leibniz algebra $(\mathfrak{h}, [\cdot, \cdot, \cdot]_T)$ on the vector space A .

Proof. for all $u_1, u_2, u_3, u_4 \in \mathfrak{h}$ and $x \in A$, we have

$$\begin{aligned} & (l_T([u_1, u_2, u_3]_T, u_4) + l_T(u_3, [u_1, u_2, u_4]_T) + l_T(u_3, u_4)l_T(u_1, u_2))(x) \\ &= [T[u_1, u_2, u_3]_T, Tu_4, x]_A + [Tu_3, T[u_1, u_2, u_4]_T, x]_A + [Tu_3, Tu_4, [Tu_1, Tu_2, x]_A]_A \\ &= [[Tu_1, Tu_2, Tu_3]_A, Tu_4, x]_A + [Tu_3, [Tu_1, Tu_2, Tu_4]_A, x]_A + [Tu_3, Tu_4, [Tu_1, Tu_2, x]_A]_A \\ &= [Tu_1, Tu_2, [Tu_3, Tu_4, x]_A]_A \\ &= l_T(u_1, u_2)([Tu_3, Tu_4, x]_A) \\ &= l_T(u_1, u_2)l_T(u_3, u_4)(x), \end{aligned}$$

which implies that (10) holds.

Then, for all $u_1, u_2, u_3, u_4 \in \mathfrak{h}$ and $x \in A$, we have

$$\begin{aligned}
 & m_T(u_3, u_4)l_T(u_1, u_2)(x) + m_T([u_1, u_2, u_3]_T, u_4)(x) + m_T(u_3, [u_1, u_2, u_4]_T)(x) \\
 &= m_T(u_3, u_4)([Tu_1, Tu_2, x]_A) + [T[u_1, u_2, u_3]_T, x, Tu_4]_A - T(\rho(T[u_1, u_2, u_3]_T, x)u_4) \\
 & \quad + [Tu_3, x, T[u_1, u_2, u_4]_T]_A - T(\rho(Tu_3, x)([u_1, u_2, u_4]_T)) \\
 & \stackrel{(29)+(16)}{=} [Tu_3, [Tu_1, Tu_2, x]_A, Tu_4]_A - T(\rho(Tu_3, [Tu_1, Tu_2, x]_A)u_4) + [[Tu_1, Tu_2, Tu_3]_A, x, Tu_4]_A \\
 & \quad - T(\rho([Tu_1, Tu_2, Tu_3]_A, x)u_4) + [Tu_3, x, [Tu_1, Tu_2, Tu_4]_A]_A \\
 & \quad - T(\rho(Tu_3, x)\rho(Tu_1, Tu_2)u_4) - T(\rho(Tu_3, x)[u_1, u_2, u_4]_{\mathfrak{h}}) \\
 & \stackrel{(1)+(2)}{=} [Tu_1, Tu_2, [Tu_3, x, Tu_4]_A]_A - T(\rho(Tu_1, Tu_2)\rho(Tu_3, x)u_4) - T([u_1, u_2, \rho(Tu_3, x)u_4]_{\mathfrak{h}}) \\
 & \stackrel{(4)+(5)}{=} [Tu_1, Tu_2, [Tu_3, x, Tu_4]_A]_A - [Tu_1, Tu_2, T(\rho(Tu_3, x)u_4)]_A \\
 & \quad = l_T(u_1, u_2)([Tu_3, x, Tu_4]_A - T(\rho(Tu_3, x)u_4)) \\
 & \stackrel{(29)}{=} l_T(u_1, u_2)m_T(u_3, u_4)(x),
 \end{aligned}$$

which implies that (11) holds, similarly, (12) holds.

For all $u_1, u_2, u_3, u_4 \in \mathfrak{h}$ and $x \in A$, we have

$$\begin{aligned}
 & r_T(u_3, u_4)m_T(u_1, u_2)(x) + m_T(u_2, u_4)m_T(u_1, u_3)(x) + l_T(u_2, u_3)m_T(u_1, u_4)(x) \\
 &= r_T(u_3, u_4)([Tu_1, x, Tu_2]_A - T(\rho(Tu_1, x)u_2)) + m_T(u_2, u_4)([Tu_1, x, Tu_3]_A - T(\rho(Tu_1, x)u_3)) \\
 & \quad + l_T(u_2, u_3)([Tu_1, x, Tu_4]_A - T(\rho(Tu_1, x)u_4)) \\
 &= [[Tu_1, x, Tu_2]_A, Tu_3, Tu_4]_A - T(\rho([Tu_1, x, Tu_2]_A, Tu_3)u_4) - [T(\rho(Tu_1, x)u_2), Tu_3, Tu_4]_A \\
 & \quad + T(\rho(T(\rho(Tu_1, x)u_2), Tu_3)u_4) + [Tu_2, [Tu_1, x, Tu_3]_A, Tu_4]_A - T(\rho(Tu_2, [Tu_1, x, Tu_3]_A)u_4) \\
 & \quad - [Tu_2, T(\rho(Tu_1, x)u_3), Tu_4]_A + T(\rho(Tu_2, T(\rho(Tu_1, x)u_3))u_4) + [Tu_2, Tu_3, [Tu_1, x, Tu_4]_A]_A \\
 & \quad - [Tu_2, Tu_3, T(\rho(Tu_1, x)u_4)]_A \\
 &= [[Tu_1, x, Tu_2]_A, Tu_3, Tu_4]_A + [Tu_2, [Tu_1, x, Tu_3]_A, Tu_4]_A + [Tu_2, Tu_3, [Tu_1, x, Tu_4]_A]_A \\
 & \quad - T(\rho(Tu_2, [Tu_1, x, Tu_3]_A)u_4) - T(\rho([Tu_1, x, Tu_2]_A, Tu_3)u_4) - [Tu_2, Tu_3, T(\rho(Tu_1, x)u_4)]_A \\
 & \stackrel{(1)}{=} [Tu_1, x, [Tu_2, Tu_3, Tu_4]_A]_A - T(\rho(Tu_2, [Tu_1, x, Tu_3]_A)u_4) - T(\rho([Tu_1, x, Tu_2]_A, Tu_3)u_4) \\
 & \quad - T(\rho(Tu_2, Tu_3)\rho(Tu_1, x)u_4) \\
 & \stackrel{(2)+(4)}{=} [Tu_1, x, T[u_2, u_3, u_4]_A]_A - T(\rho(Tu_1, x)\rho(Tu_2, Tu_3)u_4) - T(\rho(Tu_1, x)[u_2, u_3, u_4]_{\mathfrak{h}}) \\
 & \stackrel{(5)}{=} [Tu_1, x, T[u_2, u_3, u_4]_A]_A - T(\rho(Tu_1, x)\rho(Tu_2, Tu_3)u_4) \\
 & \stackrel{(16)}{=} [Tu_1, x, T[u_2, u_3, u_4]_T]_A - T(\rho(Tu_1, x)([u_2, u_3, u_4]_T)) \\
 & \quad = m_T(u_1, [u_2, u_3, u_4]_T)(x),
 \end{aligned}$$

which means that (13) holds, similarly, we can prove that (14) holds. Thus, $(A; l_T, m_T, r_T)$ is a representation of the 3-Leibniz algebra $(\mathfrak{h}, [\cdot, \cdot, \cdot]_T)$ on the vector space A .

Let $\partial_T : \mathfrak{C}_{3Leib}^n(\mathfrak{h}; A) \rightarrow \mathfrak{C}_{3Leib}^{n+1}(\mathfrak{h}; A)$, $(n \geq 1)$ be the corresponding coboundary operator of the 3-Leibniz algebra $(\mathfrak{h}, [\cdot, \cdot, \cdot]_T)$ with coefficients in the representation (A, l_T, m_T, r_T) . More precisely, $\partial_T : \mathfrak{C}_{3Leib}^n(\mathfrak{h}; A) \rightarrow \mathfrak{C}_{3Leib}^{n+1}(\mathfrak{h}; A)$ is given by

$$\begin{aligned}
 & (\partial_T \theta)(\mathfrak{U}_1, \dots, \mathfrak{U}_n, w) \\
 &= \sum_{1 \leq j < k \leq n} (-1)^j \theta(\mathfrak{U}_1, \dots, \hat{\mathfrak{U}}_j, \dots, \mathfrak{U}_{k-1}, [u_j, v_j, u_k]_T \wedge v_k, \mathfrak{U}_{k+1}, \dots, \mathfrak{U}_n, w) \\
 & \quad + \sum_{1 \leq j < k \leq n} (-1)^j \theta(\mathfrak{U}_1, \dots, \hat{\mathfrak{U}}_j, \dots, \mathfrak{U}_{k-1}, u_k \wedge [u_j, v_j, u_k]_T, \mathfrak{U}_{k+1}, \dots, \mathfrak{U}_n, w) \\
 & \quad + \sum_{j=1}^n ((-1)^j \theta(\mathfrak{U}_1, \dots, \hat{\mathfrak{U}}_j, \dots, \mathfrak{U}_n, [u_j, w]_T) + (-1)^{j+1} l_T(\mathfrak{U}_j) \theta(\mathfrak{U}_1, \dots, \hat{\mathfrak{U}}_j, \dots, \mathfrak{U}_n, w)) \\
 & \quad + (-1)^{n+1} (m_T(u_n, w) \theta(\mathfrak{U}_1, \dots, \mathfrak{U}_{n-1}, v_n) + r_T(v_n, w) \theta(\mathfrak{U}_1, \dots, \mathfrak{U}_{n-1}, u_n)).
 \end{aligned}$$

Where $\mathfrak{u}_i = u_i \wedge v_i \in \wedge^2 \mathfrak{h}$, for all $i = 1, 2, \dots, n$ and $w \in \mathfrak{h}$. Thus $(\oplus_{n=1}^{+\infty} \mathbb{C}_{3Leib}^n(\mathfrak{h}; A), \partial)$ is a cochain complex. for $\theta \in \mathbb{C}_{3Leib}^n(\mathfrak{h}; A)$.

It is obvious that $\theta \in \mathbb{C}_{3Leib}^1(\mathfrak{h}; A)$ is closed if and only if

$$\begin{aligned} & [Tu, Tv, \theta w]_A + [\theta u, Tv, Tw]_A + [Tu, \theta v, Tw]_A \\ & = T(\rho(\theta u, Tv)w + \rho(Tu, \theta v)w) + \theta(\rho(Tu, Tv)w + [u, v, w]_{\mathfrak{h}}). \end{aligned}$$

For any $\mathfrak{x} \in A \wedge A$, we define $\delta(\mathfrak{x}) : \mathfrak{h} \rightarrow A$ by

$$\delta(\mathfrak{x})v = T\rho(\mathfrak{x})v - [\mathfrak{x}, Tv]_A, \quad \forall v \in \mathfrak{h}. \quad (31)$$

Proposition 4.2. *Let T be a nonabelian embedding tensor on a 3-Lie algebra $(A, [\cdot, \cdot, \cdot]_A)$ with respect to a coherent action $(\mathfrak{h}; \rho)$. Then $\delta(\mathfrak{x})$ is a 1-cocycle on the 3-Lie algebra $(\mathfrak{h}, [\cdot, \cdot, \cdot]_T)$ with coecients in (A, l_T, m_T, r_T) .*

Proof. For any $u, v, w \in \mathfrak{h}$, we have

$$\begin{aligned} & (\partial_T \delta(\mathfrak{x}))(u, v, w) \\ & = -\delta(\mathfrak{x})([u, v, w]_T) + [Tu, Tv, \delta(\mathfrak{x})w]_A + [\delta(\mathfrak{x})u, Tv, Tw]_A \\ & \quad + [Tu, \delta(\mathfrak{x})v, Tw]_A - T(\rho(\delta(\mathfrak{x})u, Tv)w) - T(\rho(Tu, \delta(\mathfrak{x})v)w) \\ & = -T\rho(\mathfrak{x})([u, v, w]_T) + [\mathfrak{x}, T[u, v, w]_T]_A + [Tu, Tv, T\rho(\mathfrak{x})w]_A \\ & \quad - [Tu, Tv, [\mathfrak{x}, Tw]_A]_A + [T\rho(\mathfrak{x})u, Tv, Tw]_A - [[\mathfrak{x}, Tu]_A, Tv, Tw]_A \\ & \quad - T(\rho(T\rho(\mathfrak{x})u, Tv)w) + T(\rho([\mathfrak{x}, Tu]_A, Tv)w) + [Tu, T\rho(\mathfrak{x})v, Tw]_A \\ & \quad - [Tu, [\mathfrak{x}, Tv]_A, Tw]_A - T(\rho(Tu, T\rho(\mathfrak{x})v)w) + T(\rho(Tu, [\mathfrak{x}, Tv]_A)w) \\ & \stackrel{(16)+(5)+(1)}{=} -T\rho(\mathfrak{x})(\rho(Tu, Tv)w) - T\rho(\mathfrak{x})([u, v, w]_{\mathfrak{h}}) + T((\rho(Tu, Tv))\rho(\mathfrak{x})w) \\ & \quad + T(\rho(T\rho(\mathfrak{x})u, Tv)w) - T(\rho(T\rho(\mathfrak{x})u, Tv)w) + T(\rho([\mathfrak{x}, Tu]_A, Tv)w) \\ & \quad + T(\rho(Tu, T\rho(\mathfrak{x})v)w) - T(\rho(Tu, T\rho(\mathfrak{x})v)w) + T(\rho(Tu, [\mathfrak{x}, Tv]_A)w) \\ & \stackrel{(2)+(4)}{=} 0. \end{aligned}$$

Thus, we deduce that $\partial_T \delta(\mathfrak{x}) = 0$.

Now, we give a cohomology of nonabelian embedding tensors on 3-Lie algebra $(A, [\cdot, \cdot, \cdot]_A)$ with respect to the coherent action $(\mathfrak{h}; \rho)$.

Definition 4.2. Let T be a nonabelian embedding tensor on a 3-Lie algebra $(A, [\cdot, \cdot, \cdot]_A)$ with respect to a coherent action $(\mathfrak{h}; \rho)$. Define the set of n -cochains by

$$\mathbb{C}_T^n(\mathfrak{h}; A) = \begin{cases} \mathbb{C}_{3Leib}^n(\mathfrak{h}; A), & n \geq 1, \\ A \wedge A, & n = 0. \end{cases} \quad (32)$$

Define $d^T : \mathbb{C}_T^n(\mathfrak{h}; A) \rightarrow \mathbb{C}_T^{n+1}(\mathfrak{h}; A)$ by

$$d^T = \begin{cases} \partial_T, & n \geq 1, \\ \delta, & n = 0. \end{cases} \quad (33)$$

Denote the set of n -cocycles by $\mathcal{Z}_T^n(\mathfrak{h}; A)$ and the set of n -coboundaries by $\mathcal{B}_T^n(\mathfrak{h}; A)$. Denote by $\mathcal{H}_T^n(\mathfrak{h}; A) = \mathcal{Z}_T^n(\mathfrak{h}; A) / \mathcal{B}_T^n(\mathfrak{h}; A)$, $n \geq 0$ the n -th cohomology group which will be taken to be the n -th **cohomology group for the nonabelian embedding tensor T** .

Now, we give relationship between the coboundary operator d^T of the nonabelian embedding tensor and the differential l_1^T given by (23) using the Maurer-Cartan element T of the 3-Lie algebra $(C^*(\mathfrak{h}, A), \llbracket \cdot, \cdot, \cdot \rrbracket)$.

Theorem 4.3. *Let T be a nonabelian embedding tensor on the 3-Lie algebra $(A, [\cdot, \cdot, \cdot]_A)$ with respect to a coherent action $(\mathfrak{h}; \rho)$ then we have*

$$d^T \theta = (-1)^{n-1} l_1^T \theta, \quad \forall \theta \in \text{Hom}((\wedge^2 \mathfrak{h}) \otimes \cdots \otimes (\wedge^2 \mathfrak{h}) \wedge \mathfrak{h}, A), \quad n = 1, 2, \dots \quad (34)$$

Proof. For all $\mathfrak{u}_i = u_i \wedge v_i \in \wedge^2 \mathfrak{h}, i = 1, 2, \dots, n$ and $w \in \mathfrak{h}$, we have

$$\begin{aligned} & l_1(\theta)(\mathfrak{u}_1, \mathfrak{u}_2, \dots, \mathfrak{u}_n, w) \\ &= [\pi \boxplus \rho \boxplus \mu, \theta]_{3Leib}(\mathfrak{u}_1, \mathfrak{u}_2, \dots, \mathfrak{u}_n, w) \\ &= (\pi \boxplus \rho \boxplus \mu) \circ \theta - (-1)^{n-1} \theta \circ (\pi \boxplus \rho \boxplus \mu)(\mathfrak{u}_1, \dots, \mathfrak{u}_n, w) \\ &+ (\pi \boxplus \rho \boxplus \mu)(\theta(u_n \wedge \theta(\mathfrak{u}_1, \dots, \mathfrak{u}_{n-1}, v_n), w) + (\pi \boxplus \rho \boxplus \mu)(u_n \wedge \theta(\mathfrak{u}_1, \dots, \mathfrak{u}_{n-1}, v_n), w) \\ &+ (-1)^{n-1} \sum_{i=1}^n (-1)^{i-1} (\pi \boxplus \rho \boxplus \mu)(\mathfrak{u}_i, \theta(\mathfrak{u}_1, \dots, \hat{\mathfrak{u}}_i, \dots, \mathfrak{u}_n, w)) \\ &- (-1)^{n-1} \sum_{k=1}^{n-1} \sum_{i=1}^k (-1)^{i+1} \theta(\mathfrak{u}_1, \dots, \hat{\mathfrak{u}}_i, \dots, \mathfrak{u}_k, (\pi \boxplus \rho \boxplus \mu)(\mathfrak{u}_i, u_{k+1}) \wedge v_{k+1}, \mathfrak{u}_{k+2}, \dots, \mathfrak{u}_n, w) \\ &- (-1)^{n-1} \sum_{k=1}^{n-1} \sum_{i=1}^k (-1)^{i+1} \theta(\mathfrak{u}_1, \dots, \hat{\mathfrak{u}}_i, \dots, \mathfrak{u}_k, u_{k+1} \wedge (\pi \boxplus \rho \boxplus \mu)(\mathfrak{u}_i, w), \mathfrak{u}_{k+2}, \dots, \mathfrak{u}_n, w) \\ &- (-1)^{n-1} \sum_{i=1}^n (-1)^{i+1} \theta(\mathfrak{u}_1, \dots, \hat{\mathfrak{u}}_i, \dots, \mathfrak{u}_n, (\pi \boxplus \rho \boxplus \mu)(\mathfrak{u}_i, w)) \\ &= - (-1)^{n-1} \sum_{k=1}^{n-1} \sum_{i=1}^k (-1)^{i+1} \theta(\mathfrak{u}_1, \dots, \hat{\mathfrak{u}}_i, \dots, \mathfrak{u}_k, \mu(u_i, v_i, u_{k+1}) \wedge v_{k+1}, \mathfrak{u}_{k+2}, \dots, \mathfrak{u}_n, w) \\ &- (-1)^{n-1} \sum_{k=1}^{n-1} \sum_{i=1}^k (-1)^{i+1} \theta(\mathfrak{u}_1, \dots, \hat{\mathfrak{u}}_i, \dots, \mathfrak{u}_k, u_{k+1} \wedge \mu(u_i, v_i, w), \mathfrak{u}_{k+2}, \dots, \mathfrak{u}_n, w) \\ &- (-1)^{n-1} \sum_{i=1}^n (-1)^{i+1} \theta(\mathfrak{u}_1, \dots, \hat{\mathfrak{u}}_i, \dots, \mathfrak{u}_n, \mu(u_i, v_i, w)), \end{aligned}$$

by lemma (3.4) we have

$$\begin{aligned} & 2l_1^T \theta(\mathfrak{u}_1, \mathfrak{u}_2, \dots, \mathfrak{u}_n, w) \\ &= [[T, T, \theta]](\mathfrak{u}_1, \mathfrak{u}_2, \dots, \mathfrak{u}_n, w) \\ &= [[[\pi \boxplus \rho \boxplus \mu, T]_{3Leib}, T]_{3Leib}, \theta]_{3Leib}(\mathfrak{u}_1, \mathfrak{u}_2, \dots, \mathfrak{u}_n, w) \\ &= [[[\pi \boxplus \rho \boxplus \mu, T]_{3Leib}, T]_{3Leib}, \theta]_{3Leib}(\theta(\mathfrak{u}_1, \dots, \mathfrak{u}_{n-1}, \mathfrak{u}_n) \wedge v_n, w) \\ &+ [[[\pi \boxplus \rho \boxplus \mu, T]_{3Leib}, T]_{3Leib}, \theta]_{3Leib}(u_n \wedge \theta(\mathfrak{u}_1, \dots, \mathfrak{u}_{n-1}, v_n), w) \\ &+ (-1)^{n-1} \sum_{i=1}^n (-1)^{i-1} [[[\pi \boxplus \rho \boxplus \mu, T]_{3Leib}, T]_{3Leib}, \theta]_{3Leib}(\mathfrak{u}_i, \theta(\mathfrak{u}_1, \dots, \hat{\mathfrak{u}}_i, \dots, \mathfrak{u}_n, w)) \\ &- (-1)^{n-1} \sum_{k=1}^{n-1} \sum_{i=1}^k (-1)^{i+1} \theta(\mathfrak{u}_1, \dots, \hat{\mathfrak{u}}_i, \dots, \mathfrak{u}_k, [[[\pi \boxplus \rho \boxplus \mu, T]_{3Leib}, T]_{3Leib}, \theta]_{3Leib}(\mathfrak{u}_i, u_{k+1}) \wedge v_{k+1}, \\ &\quad \mathfrak{u}_{k+2}, \dots, \mathfrak{u}_n, w)) \\ &- (-1)^{n-1} \sum_{k=1}^{n-1} \sum_{i=1}^k (-1)^{i+1} \theta(\mathfrak{u}_1, \dots, \hat{\mathfrak{u}}_i, \dots, \mathfrak{u}_k, u_{k+1} \wedge [[[\pi \boxplus \rho \boxplus \mu, T]_{3Leib}, T]_{3Leib}, \theta]_{3Leib}(\mathfrak{u}_i, u_{k+1}), \\ &\quad \mathfrak{u}_{k+2}, \dots, \mathfrak{u}_n, w)) \\ &- (-1)^{n-1} \sum_{i=1}^n (-1)^{i+1} \theta(\mathfrak{u}_1, \dots, \hat{\mathfrak{u}}_i, \dots, \mathfrak{u}_n, [[[\pi \boxplus \rho \boxplus \mu, T]_{3Leib}, T]_{3Leib}, \theta]_{3Leib}(\mathfrak{u}_i, w)) \\ &= m_T(u_n, w) \theta(\mathfrak{u}_1, \dots, u_n) + r_T(v_n, w) \theta(\mathfrak{u}_1, \dots, v_n) \\ &+ (-1)^{n-1} \sum_{i=1}^n (-1)^{i-1} l_T(u_i, v_i) \theta(\mathfrak{u}_1, \dots, \hat{\mathfrak{u}}_i, \dots, \mathfrak{u}_n, w) \\ &- (-1)^{n-1} \sum_{k=1}^{n-1} \sum_{i=1}^k (-1)^{i+1} \theta(\mathfrak{u}_1, \dots, \hat{\mathfrak{u}}_i, \dots, \mathfrak{u}_k, \rho(Tu_i, Tv_i) u_{k+1} \wedge v_{k+1}, \mathfrak{u}_{k+2}, \dots, \mathfrak{u}_n, w) \end{aligned}$$

$$\begin{aligned}
& -(-1)^{n-1} \sum_{k=1}^{n-1} \sum_{i=1}^k (-1)^{i+1} \theta(\mathfrak{U}_1, \dots, \hat{\mathfrak{U}}_i, \dots, \mathfrak{U}_k, \mathfrak{u}_{k+1} \wedge \rho(T\mathfrak{u}_i, T\mathfrak{v}_i)\mathfrak{u}_{k+1}, \mathfrak{U}_{k+2}, \dots, \mathfrak{U}_n, w) \\
& -(-1)^{n-1} \sum_{i=1}^n (-1)^{i+1} \theta(\mathfrak{U}_1, \dots, \hat{\mathfrak{U}}_i, \dots, \mathfrak{U}_n, \rho(T\mathfrak{u}_i, T\mathfrak{v}_i)w).
\end{aligned}$$

Then we deduce that $d^T \theta = (-1)^{n-1}(l_1(\theta) + 2l_3[[T, T, \theta]])$, which implies that $d^T \theta = (-1)^{n-1}l_1^T \theta$. The proof is completed.

We can use these cohomology groups to characterize infinitesimal deformations of nonabelian embedding tensors on 3-Lie algebras in the following section.

5. Infinitesimal Deformations of Nonabelian Embedding Tensors on 3-Lie algebras

In this section, we study infinitesimal deformations of a nonabelian embedding tensor T on a 3-Lie algebra using the cohomology theory given in the previous section. In particular, we introduce the notion of a Nijenhuis element associated to a nonabelian embedding tensor T on a 3-Lie algebra, which leads to a trivial infinitesimal deformation of the nonabelian embedding tensors T a 3-Lie algebra. Also we study their relationship with the infinitesimal deformations of the associated descendent 3-Leibniz algebra.

Definition 5.1. Let T be a nonabelian embedding tensor on a 3-Lie algebra A with respect to a coherent action $(\mathfrak{h}; \rho)$ and $T : \mathfrak{h} \rightarrow A$ a linear map. If $T_t = T + t\mathfrak{T}$ is still a nonabelian embedding tensor on the Lie algebra A with respect to the coherent action $(\mathfrak{h}; \rho)$ for all $t \in \mathbb{C}$, we say that T generates a **(one-parameter) infinitesimal deformation** of the nonabelian embedding tensor T .

It is direct to deduce that $T_t = T + t\mathfrak{T}$ is an infinitesimal deformation of a nonabelian embedding tensor T if and only if for any $u, v, w \in \mathfrak{h}$

$$[\mathfrak{T}u, \mathfrak{T}v, \mathfrak{T}w]_A = \mathfrak{T}(\rho(\mathfrak{T}u, \mathfrak{T}v)w) \quad (35)$$

$$\begin{aligned}
& [\mathfrak{T}u, Tv, Tw]_A + [Tu, \mathfrak{T}v, Tw]_A + [Tu, Tv, \mathfrak{T}w]_A \\
& = T(\rho(\mathfrak{T}u, Tv)w + \rho(Tu, \mathfrak{T}v)w) + \mathfrak{T}(\rho(Tu, Tv)w + [u, v, w]_{\mathfrak{h}}).
\end{aligned} \quad (36)$$

Note that Eq (36) means that \mathfrak{T} is a 2-cocycle of the nonabelian embedding tensor T . Hence, \mathfrak{T} defines a cohomology class in $\mathcal{H}_T^2(\mathfrak{h}; A)$.

Let $(\mathcal{L}, [\cdot, \cdot, \cdot]_{\mathcal{L}})$ be a 3-Leibniz algebra and $\phi : \mathcal{L} \wedge \mathcal{L} \wedge \mathcal{L} \rightarrow \mathcal{L}$ be a linear map. If for any $t \in \mathbb{C}$, the multiplication $[\cdot, \cdot, \cdot]_T$ defined by

$$[u, v, w]_T = [u, v, w] + t(u, v, w), \quad \forall u, v, w \in \mathcal{L},$$

also gives a 3-Leibniz algebra structure, we say that ϕ generates an infinitesimal deformation of the 3-Leibniz algebra $(\mathcal{L}, [\cdot, \cdot, \cdot]_{\mathcal{L}})$. The two types of infinitesimal deformations are related as follows

Proposition 5.1. *If T generates an infinitesimal deformation of a nonabelian embedding tensor T on a 3-Lie algebra A with respect to a coherent action $(\mathfrak{h}; \rho)$, then the product $\phi_{\mathfrak{T}}$ on \mathfrak{h} defined by*

$$\phi_{\mathfrak{T}}(u, v, w) = \rho(Tu, Tv), \quad \forall u, v, w \in \mathfrak{h},$$

generates an infinitesimal deformation of the associated 3-Leibniz algebra $(\mathfrak{h}, [\cdot, \cdot, \cdot]_T)$

Proof. We denote by $[\cdot, \cdot, \cdot]_t$ the corresponding 3-Leibniz algebra structure associated to the nonabelian embedding tensor $T + t\mathfrak{T}$. Then we have

$$\begin{aligned}
[u, v, w]_t &= \rho((T + tT)u, (T + tT)v)w + [u, v, w]_{\mathfrak{h}} \\
&= \rho(Tu, Tv)w + [u, v, w]_{\mathfrak{h}} + t\rho(\mathfrak{T}u, \mathfrak{T}v)w \\
&= [u, v, w]_T + t\phi_{\mathfrak{T}}(u, v, w), \quad \forall u, v, w \in \mathfrak{h},
\end{aligned}$$

which implies that $\phi_{\mathfrak{T}}$ generates an infinitesimal deformation of $(\mathfrak{h}, [\cdot, \cdot, \cdot]_T)$

Definition 5.2. Let T be a nonabelian embedding tensor on a 3-Lie algebra A with respect to a coherent action

$(\mathfrak{h}; \rho)$ Two infinitesimal deformations $T_t^1 = T + t\mathfrak{T}_1$ and $T_t^2 = T + t\mathfrak{T}_2$ are called **equivalent** if there exists an element $\mathfrak{x} \in \wedge^2 A$ such that $(\text{Id}_A + t\text{ad}_{\mathfrak{x}}, \text{Id}_{\mathfrak{h}} + t\rho(\mathfrak{x}))$ is a homomorphism from T_t^2 to T_t^1 . In particular, an infinitesimal deformation $T_t = T + t\mathfrak{T}$ of a nonabelian embedding tensor T is called trivial if there exists an element $\mathfrak{x} \in \wedge^2 A$ such that $(\text{Id}_A + t\text{ad}_{\mathfrak{x}}, \text{Id}_{\mathfrak{h}} + t\rho(\mathfrak{x}))$ is a homomorphism from T_t to T .

Let $(\text{Id}_A + t\text{ad}_{\mathfrak{x}}, \text{Id}_{\mathfrak{h}} + t\rho(\mathfrak{x}))$ be a homomorphism from T_t^2 to T_t^1 . Then, by we have

$$(\text{Id}_A + t\text{ad}_{\mathfrak{x}})(T + t\mathfrak{T}_1) = (T + t\mathfrak{T}_2)(\text{Id}_{\mathfrak{h}} + t\rho(\mathfrak{x}))$$

thus we have

$$\begin{aligned} (\text{Id}_A + t\text{ad}_{\mathfrak{x}})[y, z, t]_A &= [(\text{Id}_A + t\text{ad}_{\mathfrak{x}})(y), (\text{Id}_A + t\text{ad}_{\mathfrak{x}})(z), (\text{Id}_A + t\text{ad}_{\mathfrak{x}})(t)]_A, \forall x, y, z \in A, \\ (\text{Id}_{\mathfrak{h}} + t\text{ad}_{\mathfrak{x}})[u, v, w]_{\mathfrak{h}} &= [(\text{Id}_{\mathfrak{h}} + t\text{ad}_{\mathfrak{x}})(u), (\text{Id}_{\mathfrak{h}} + t\text{ad}_{\mathfrak{x}})(v), (\text{Id}_{\mathfrak{h}} + t\text{ad}_{\mathfrak{x}})(w)]_{\mathfrak{h}}, \forall u, v, w \in \mathfrak{h}, \end{aligned}$$

which implies that \mathfrak{x} satisfies

$$[[\mathfrak{x}, x]_A, [\mathfrak{x}, y]_A, [\mathfrak{x}, z]_A]_A = 0, \quad \forall x, y, z \in A. \quad (37)$$

Then by Eq (6), we get

$$(T + t\mathfrak{T}_1)(\text{Id}_{\mathfrak{h}} + t\rho(\mathfrak{x}))u = (\text{Id}_A + t\text{ad}_{\mathfrak{x}})(T + t\mathfrak{T}_2)(u), \quad \forall u \in \mathfrak{h},$$

which implies

$$\begin{cases} (\mathfrak{T}_1 - \mathfrak{T}_2)(u) &= T\rho(\mathfrak{x})u - [\mathfrak{x}, Tu]_A, \\ [\mathfrak{x}, \mathfrak{T}_2 u]_A &= \mathfrak{T}_1\rho(\mathfrak{x})(u), \quad \forall u \in \mathfrak{h}. \end{cases} \quad (38)$$

Finally, Eq (7) gives

$$(\text{Id}_{\mathfrak{h}} + t\rho(\mathfrak{x}))\rho(x, y)(u) = \rho((\text{Id}_A + t\text{ad}_{\mathfrak{x}})(x, y))(\text{Id}_{\mathfrak{h}} + t\rho(\mathfrak{x}))(u), \quad \forall x, y \in A, u \in \mathfrak{h},$$

which implies that \mathfrak{x} satisfies

$$\rho([\mathfrak{x}, x]_A, y)\rho(\mathfrak{x}) = 0, \quad \forall \mathfrak{x} \in \wedge^2 A \text{ and } y \in A. \quad (39)$$

Next, we have the main result in this section:

Theorem 5.2. *Let T be a nonabelian embedding tensor on a 3-Lie algebra A with respect to a coherent action $(\mathfrak{h}; \rho)$. If two infinitesimal deformations $T_t^1 = T + t\mathfrak{T}_1$ and $T_t^2 = T + t\mathfrak{T}_2$ are equivalent, then \mathfrak{T}_1 and \mathfrak{T}_2 are in the same cohomology class of $\mathcal{H}^2(T)$.*

Proof. It easy to observe from the first condition of (38) that

$$\mathfrak{T}_1(u) = \mathfrak{T}_2(u) + (\partial\mathfrak{x})(u), \quad \forall u \in \mathfrak{h},$$

which implies that \mathfrak{T}_1 and \mathfrak{T}_2 are in the same cohomology class.

Definition 5.3. Let T be a nonabelian embedding tensor on a 3-Lie algebra A with respect to a coherent action $(\mathfrak{h}; \rho)$. An element $\mathfrak{x} \in \wedge^2 A$ is said to be a **Nijenhuis element** associated to T if \mathfrak{x} satisfies

$$\begin{aligned} [[\mathfrak{x}, x]_A, [\mathfrak{x}, y]_A, [\mathfrak{x}, z]_A]_A &= 0, \\ \rho([\mathfrak{x}, x]_A, y)\rho(\mathfrak{x}) &= 0, \\ [\mathfrak{x}, T\rho(\mathfrak{x})u - [\mathfrak{x}, Tu]_A]_A &= 0, \end{aligned} \quad (40)$$

for all $x, y, z \in A$ and $u \in \mathfrak{h}$.

Denote by $\text{Nij}(T)$ the set of Nijenhuis elements associated to a nonabelian embedding tensor T

It is obvious that a trivial infinitesimal deformation leads to a Nijenhuis element. Conversely, a Nijenhuis element can also generate a trivial infinitesimal deformation as the following theorem shows.

Theorem 5.3. *Let T be a nonabelian embedding tensor on a 3-Lie algebra A with respect to a coherent action $(\mathfrak{h}; \rho)$. Then for any $\mathfrak{X} \in \text{Nij}(T)$, $T_t = T + t\mathfrak{X}$ with $\mathfrak{X} = \partial_T \mathfrak{X}$ is a trivial infinitesimal deformation of the nonabelian embedding tensor T .*

We need the following lemma to prove this theorem.

Lemma 5.4. *Let T be a nonabelian embedding tensor on a 3-Lie algebra A with respect to a coherent action $(\mathfrak{h}; \rho)$. Let $\psi_A : A \rightarrow A$ and $\psi_{\mathfrak{h}} : \mathfrak{h} \rightarrow \mathfrak{h}$ be two 3-Lie algebra isomorphisms such that Eq. (6) and Eq. (7) hold. Then $\psi_A^{-1} \circ T \circ \psi_{\mathfrak{h}}$ is a nonabelian embedding tensor on the 3-Lie algebra A with respect to the coherent action $(\mathfrak{h}; \rho)$.*

Proof. It follows from straightforward computations.

The proof of Theorem (5.3): For any Nijenhuis element $\mathfrak{X} \in \text{Nij}(T)$, we define a linear map $\mathfrak{X} : \mathfrak{h} \rightarrow A$ by

$$\mathfrak{X}(u) = (\partial_T \mathfrak{X})(u) = T\rho(\mathfrak{X})u - [\mathfrak{X}, Tu]_A, \quad \forall u \in \mathfrak{h}. \quad (41)$$

Let $T_t = T + t\mathfrak{X}$. By the definition of Nijenhuis elements, for all $x, y, z \in A, u, v, w \in \mathfrak{h}$, we have

$$\begin{aligned} (\text{Id}_A + t\text{ad}_{\mathfrak{X}})[x, y, z]_A &\stackrel{(37)}{=} [(\text{Id}_A + t\text{ad}_{\mathfrak{X}})(x), (\text{Id}_A + t\text{ad}_{\mathfrak{X}})(y), (\text{Id}_A + t\text{ad}_{\mathfrak{X}})(z)]_A, \\ (\text{Id}_{\mathfrak{h}} + t\rho(\mathfrak{X}))[u, v, w]_{\mathfrak{h}} &\stackrel{(4)}{=} [(\text{Id}_{\mathfrak{h}} + t\rho(\mathfrak{X}))(u), (\text{Id}_{\mathfrak{h}} + t\rho(\mathfrak{X}))(v), (\text{Id}_{\mathfrak{h}} + t\rho(\mathfrak{X}))(w)]_{\mathfrak{h}}, \\ (\text{Id}_A + t\text{ad}_{\mathfrak{X}}) \circ T &\stackrel{(40)}{=} T \circ (\text{Id}_{\mathfrak{h}} + t\rho(\mathfrak{X})), \\ (\text{Id}_{\mathfrak{h}} + t\rho(\mathfrak{X}))\rho(x, y)u &\stackrel{(39)}{=} \rho((\text{Id}_A + t\text{ad}_{\mathfrak{X}})x)(\text{Id}_{\mathfrak{h}} + t\rho(\mathfrak{X}))(y), (\text{Id}_A + t\text{ad}_{\mathfrak{X}})y)(\text{Id}_{\mathfrak{h}} + t\rho(\mathfrak{X}))(u). \end{aligned}$$

Since $\text{ad}_{\mathfrak{X}}$ and $\rho(\mathfrak{X})$ are linear transformations of finite-dimensional \mathbb{C} -vector spaces, it follows that $\text{Id}_A + t\text{ad}_{\mathfrak{X}}$ and $\text{Id}_{\mathfrak{h}} + t\rho(\mathfrak{X})$ are 3-Lie algebra isomorphisms for $|t|$ sufficiently small. Thus, by Lemma (5.4)

$$\begin{aligned} &(\text{Id}_A + t\text{ad}_{\mathfrak{X}})^{-1} \circ T \circ T \circ (\text{Id}_{\mathfrak{h}} + t\rho(\mathfrak{X})) \\ &= \sum_{i=0}^{+\infty} (-t\text{ad}_{\mathfrak{X}})^i \circ (T + tT\rho(\mathfrak{X})) \\ &= T + t(T\rho(\mathfrak{X}) - \text{ad}_{\mathfrak{X}} \circ T) + \sum_{i=0}^{+\infty} (-1)^i t^{i+1} \text{ad}_{\mathfrak{X}}^i \circ (T\rho(\mathfrak{X}) - \text{ad}_{\mathfrak{X}} \circ T) \\ &\stackrel{(40)}{=} T + t(T\rho(\mathfrak{X}) - \text{ad}_{\mathfrak{X}} \circ T) = T + t\mathfrak{X} = T_t \end{aligned}$$

is a nonabelian embedding tensor on the 3-Lie algebra $(A, [\cdot, \cdot, \cdot]_A)$ with respect to the coherent action $(\mathfrak{h}; \rho)$ for $|t|$ sufficiently small. Thus, T_t given by Eq. (41) satisfies the conditions (35) and (36). Therefore, T_t is a nonabelian embedding tensor for all t , which means that T given by Eq. (41) generates an infinitesimal deformation of T . It is straightforward to show that this infinitesimal deformation is trivial.

Now we recall the notion of a Nijenhuis operator on a 3-Leibniz algebra given in [37] which leads to a trivial infinitesimal deformation of a 3-Leibniz algebra.

Definition 5.4. A linear map $N : \mathcal{L} \rightarrow \mathcal{L}$ on a 3-Leibniz algebra $(\mathcal{L}, [\cdot, \cdot, \cdot]_{\mathcal{L}})$ is said to be a **Nijenhuis operator** if

$$\begin{aligned} [Nu, Nv, Nw]_{\mathcal{L}} &= N([Nu, Nv, w]_{\mathcal{L}} + [u, Nv, Nw]_{\mathcal{L}} + [Nu, v, Nw]_{\mathcal{L}}) \\ &\quad - N[Nu, v, w]_{\mathcal{L}} - N[u, Nv, w]_{\mathcal{L}} - N[u, v, Nw]_{\mathcal{L}} \\ &\quad + N^2[u, v, w]_{\mathcal{L}}, \quad \forall u, v, w \in \mathcal{L}. \end{aligned} \quad (42)$$

For its relationship with a Nijenhuis element associated to a nonabelian embedding tensor, we have the following result.

Proposition 5.5. *Let $\mathfrak{X} \in \wedge^2 A$ be a Nijenhuis element associated to a nonabelian embedding tensor T on a 3-Lie algebra A with respect to a coherent action $(\mathfrak{h}; \rho)$. Then $\rho(\mathfrak{X})$ is a Nijenhuis operator on the descendent 3-Leibniz algebra $(\mathfrak{h}, [\cdot, \cdot, \cdot]_T)$.*

Proof. For all $u, v, w \in \mathfrak{h}$, we have

$$\begin{aligned}
 & \rho(\mathfrak{X})\left([\rho(\mathfrak{X})u, \rho(\mathfrak{X})v, w]_T + [u, \rho(\mathfrak{X})v, \rho(\mathfrak{X})w]_T + [\rho(\mathfrak{X})u, v, \rho(\mathfrak{X})w]_T - \rho(\mathfrak{X})([\rho(\mathfrak{X})u, v, w]_T)\right. \\
 & \quad \left. - \rho(\mathfrak{X})([u, \rho(\mathfrak{X})v, w]_T) - \rho(\mathfrak{X})([u, v, \rho(\mathfrak{X})w]_T) + \rho^2(\mathfrak{X})([u, v, w]_T)\right) - [\rho(\mathfrak{X})u, \rho(\mathfrak{X})v, \rho(\mathfrak{X})w]_T \\
 \stackrel{(16)}{=} & \rho(\mathfrak{X})\left(\rho(T\rho(\mathfrak{X})u, T\rho(\mathfrak{X})v)w + [\rho(\mathfrak{X})u, \rho(\mathfrak{X})v, w]_{\mathfrak{h}} + \rho(Tu, T\rho(\mathfrak{X})v)\rho(\mathfrak{X})w + [u, \rho(\mathfrak{X})v, \rho(\mathfrak{X})w]_{\mathfrak{h}}\right. \\
 & \quad \left. + \rho(T\rho(\mathfrak{X})u, T\rho(\mathfrak{X})v)w + [\rho(\mathfrak{X})u, \rho(\mathfrak{X})v, w]_{\mathfrak{h}} - \rho(\mathfrak{X})\rho(T\rho(\mathfrak{X})u, Tv)w - \rho(\mathfrak{X})[\rho(\mathfrak{X})u, v, w]_{\mathfrak{h}}\right. \\
 & \quad \left. - \rho(\mathfrak{X})\rho(Tu, T\rho(\mathfrak{X})v)w - \rho(\mathfrak{X})[u, \rho(\mathfrak{X})v, w]_{\mathfrak{h}} - \rho(\mathfrak{X})\rho(Tu, Tv)\rho(\mathfrak{X})w - \rho(\mathfrak{X})[u, v, \rho(\mathfrak{X})w]_{\mathfrak{h}}\right. \\
 & \quad \left. + \rho^2(\mathfrak{X})\rho(Tu, Tv)w + \rho^2(\mathfrak{X})[u, v, w]_{\mathfrak{h}}\right) - \rho(T\rho(\mathfrak{X})u, T\rho(\mathfrak{X})v)\rho(\mathfrak{X})w - [\rho(\mathfrak{X})u, \rho(\mathfrak{X})v, \rho(\mathfrak{X})w]_{\mathfrak{h}} \\
 \stackrel{(4)}{=} & \rho(\mathfrak{X})\left(\rho(T\rho(\mathfrak{X})u, T\rho(\mathfrak{X})v)w + \rho(Tu, T\rho(\mathfrak{X})v)\rho(\mathfrak{X})w + \rho(T\rho(\mathfrak{X})u, Tv)\rho(\mathfrak{X})w\right. \\
 & \quad \left. - \rho(\mathfrak{X})(\rho(T\rho(\mathfrak{X})u, Tv)w - \rho(\mathfrak{X})(\rho(Tu, T\rho(\mathfrak{X})v)w - \rho(\mathfrak{X})(\rho(Tu, Tv)\rho(\mathfrak{X})w\right. \\
 & \quad \left. + \rho^2(\mathfrak{X})\rho(Tu, Tv)w)\right) - \rho(T\rho(\mathfrak{X})u, T\rho(\mathfrak{X})v)\rho(\mathfrak{X})w \\
 = & \rho([\mathfrak{X}, T\rho(\mathfrak{X})u]_A, T\rho(\mathfrak{X})v) + \rho(T\rho(\mathfrak{X})u, [\mathfrak{X}, T\rho(\mathfrak{X})v]_A) - \rho(\mathfrak{X})\rho([\mathfrak{X}, Tu]_A, T\rho(\mathfrak{X})v) \\
 & \quad - \rho(\mathfrak{X})\rho(Tu, [\mathfrak{X}, T\rho(\mathfrak{X})v]_A) - \rho(\mathfrak{X})\rho([\mathfrak{X}, T\rho(\mathfrak{X})u]_A, Tv) - \rho(\mathfrak{X})\rho(T\rho(\mathfrak{X})u, [\mathfrak{X}, Tv]_A) \\
 & \quad + \rho^2(\mathfrak{X})\rho([\mathfrak{X}, Tu]_A, Tv) + \rho^2(\mathfrak{X})\rho(Tu, [\mathfrak{X}, Tv]_A) \\
 \stackrel{(40)}{=} & \rho([\mathfrak{X}, [\mathfrak{X}, Tu]_A]_A, T\rho(\mathfrak{X})v) + \rho(T\rho(\mathfrak{X})u, [\mathfrak{X}, [\mathfrak{X}, Tv]_A]_A) - \rho(\mathfrak{X})\rho([\mathfrak{X}, Tu]_A, T\rho(\mathfrak{X})v) \\
 & \quad - \rho(\mathfrak{X})\rho(Tu, [\mathfrak{X}, [\mathfrak{X}, Tv]_A]_A) - \rho(\mathfrak{X})\rho([\mathfrak{X}, [\mathfrak{X}, Tu]_A]_A, Tv) - \rho(\mathfrak{X})\rho(T\rho(\mathfrak{X})u, [\mathfrak{X}, Tv]_A) \\
 & \quad + \rho^2(\mathfrak{X})\rho([\mathfrak{X}, Tu]_A, Tv) + \rho^2(\mathfrak{X})\rho(Tu, [\mathfrak{X}, Tv]_A) \\
 \stackrel{(39)}{=} & 0.
 \end{aligned}$$

Then, $\rho(\mathfrak{X})$ is a Nijenhuis operator on the 3-Leibniz algebra $(\mathfrak{h}, [\cdot, \cdot, \cdot]_T)$.

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