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Formulation and Simplification of ECT Spin-Gravitational Field Theory in a Given Riemannian Background Spacetime

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Abstract
This paper is an improved version of the thesis 【5】.The ECT Spin-Gravitational Field theory is the promotion of the general relativity (Einstein gravitational field theory) in curved spacetime with torsion,it eliminates the contradiction between the Einstein gravitational field theory and the Dirac electron field theory.The simplified form of the ECT spin-gravity theory has a limit of Einstein gravitational field theory,so the ECT spin-gravitational theory should be more correct than the Einstein gravitational field theory. In order to obtaining a clear physical picture and analyzing a specific physical problem, the given Riemann background presentation of the ECT spin-gravitional field has been established through separating the physical effect of spin-gravitional field from the given Riemann background spacetime in this article. In this article,the gauge conditions and simplified form of the ECT spin-gravitational field has been discussed in a given background spacetime.

Keywords: General Relativity, ECT Spin-Gravitational Field Theory, Background Picture, Gauge Conditions

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Introduction

There is an irreconcilable contradiction between Einstein's gravitational field theory and Dirac's electron field theory [1]. We can eliminate this contradiction by extending Einstein's gravitational field theory to ECT spin-gravitational field theory with curved space-time [2]. Since the ECT spin-gravity field theory in its simplified form has the Einstein gravitational field theory limit, we believe that the ECT spin-gravity field theory is a more correct gravitational theory than the Einstein gravitational field theory. The force field in the ECT spin-gravitational field theory is a generalized gravitational field: the spin-gravitational field, which is partially equivalent to the gravitational field in Einstein's gravitational field theory.

In the standard ECT spin-gravitational field theory, the physical effects of gravitational field and spin field are characterized by the frame field and frame affine connection describing the space-time structure, respectively. The physical effects of gravitational field and spin field are manifested as the change of space-time structure, and their mathematical picture is very clear, but the physical picture is not clear enough [2]. It is well known that the physical picture is very clear in Newtonian mechanics, special relativity, classical electrodynamics and quantum mechanics. The fundamental reason is that the motion of matter is assumed to be in a given flat background space-time, and the motion of matter does not change the space-time structure of the flat background space-time. Therefore, in order to get a clear physical picture in the ECT spin-gravity field theory, we assume that the spin-gravity field is a complex of the geometric structure of the given background space-time and the dynamic field under the given background space-time, so the motion of matter not only has an impact on the space-time structure of the given background spacetime, but also has an impact on the dynamic field of the spin-gravity field. Only the flat background space-time is not affected by the motion of matter.

In this paper ,we believe: In the ECT spin-gravitation field theory, the physical effects of the spin field and the gravitation field are described by the deviation physical quantities of the curved spacetime with torsion to the specified Riemannian background spacetime, and the spin field and the gravitation field can be essentially decomposed into the geometry of the given Riemannian background spacetime and the deviation physical quantities in the given Riemannian background spacetime. In this paper, we discuss the formulation and gauge conditions of the ECT spin-gravity field theory in a simplified form on a given Riemannian background spacetime.

Introduction to Spacetime Geometry

The real physical space-time is the curved space-time with torsion. In order to study the real physical space-time in the specified background space-time, we must have a certain understanding of the geometry of the given background space-time and the curved space-time with torsion. This section briefly introduces some geometric properties of the given background spacetime and the curved spacetime with torsion, and uses the definition of the coupling differential to connect the given background spacetime and the curved spacetime with torsion.

Geometry of Riemannian Background Spacetime

Let $\tilde{\lambda}^{(\alpha)}_{\mu},\tilde{\lambda}^{\mu}_{(\alpha)}$ is the tetrad field in a given Riemannian background spacetime $\tilde{M}^4$, $\tilde{g}_{\mu\nu},\tilde{g}^{\mu\nu}$ is the coordinate metric tensor in a given Riemannian background spacetime $\tilde{M}^4$, $\eta_{(\alpha\beta)},\eta^{(\alpha\beta)}$ is the tetrad metric tensor in a given Riemann background spacetime $\tilde{M}^4$, $\delta^{\nu}_{\mu},\delta^{(\alpha)}_{(\beta)}$ is the unit coordinate tensor and the unit tetrad tensor in the given Riemann background spacetime $\tilde{M}^4$. In the given Riemann background space-time $\tilde{M}^4$, Coordinate indicators use $\tilde{g}_{\mu\nu},\tilde{g}^{\mu\nu}$ to rise and fall ,tetrad indicators use $\eta_{(\alpha\beta)},\eta^{(\alpha\beta)}$ to rise and fall, Using $\tilde{\lambda}^{(\alpha)}_{\mu},\tilde{\lambda}^{\mu}_{(\alpha)}$ as the transformation between tetrad indicators and coordinate indicators, there are the following relations:

$$\begin{aligned}
&\tilde{\lambda}^{(\alpha)}_{\mu}\tilde{\lambda}^{\nu}_{(\alpha)}=\delta^{\nu}_{\mu},\tilde{\lambda}^{(\alpha)}_{\rho}\tilde{\lambda}^{\rho}_{(\beta)}=\delta^{(\alpha)}_{(\beta)}\\
&\tilde{\lambda}_{(\alpha)\mu}=\eta_{(\alpha\beta)}\tilde{\lambda}^{(\beta)}_{\mu}=\tilde{g}_{\mu\nu}\tilde{\lambda}^{\nu}_{(\alpha)}\\
&\tilde{\lambda}^{(\alpha)\mu}=\eta^{(\alpha\beta)}\tilde{\lambda}^{\mu}_{(\beta)}=\tilde{g}^{\mu\nu}\tilde{\lambda}^{(\alpha)}_{\nu}\\
&\eta_{(\alpha\beta)}=\tilde{\lambda}^{\mu}_{(\alpha)}\tilde{\lambda}_{(\beta)\mu},\eta^{(\alpha\beta)}=\tilde{\lambda}^{(\alpha)\mu}\tilde{\lambda}^{(\beta)}_{\mu}\\
&\tilde{g}_{\mu\nu}=\tilde{\lambda}^{(\alpha)}_{\mu}\tilde{\lambda}_{(\alpha)\nu},\tilde{g}^{\mu\nu}=\tilde{\lambda}^{(\alpha)\mu}\tilde{\lambda}^{\nu}_{(\alpha)}
\end{aligned} \tag{2.1}$$

where:

$$\eta_{(\alpha\beta)}=\begin{pmatrix}1 & 0\\ 0 & -\delta_{(ab)}\end{pmatrix},\eta^{(\alpha\beta)}=\begin{pmatrix}1 & 0\\ 0 & -\delta^{(ab)}\end{pmatrix}$$
$$\alpha,\beta=0,1,2,3;a,b=1,2,3$$

Let $\tilde{\Gamma}^{\rho}_{\sigma\mu},\tilde{\Gamma}^{(\alpha)}_{(\beta)\mu},\tilde{\Gamma}_{\mu}$ is the coordinate, tetrad, and spinor affine connections in a given Riemannian background spacetime, we can define the basic covariant derivative in a given Riemannian background spacetime:

（1）For the scalar ϕ：

$$\tilde{D}_{\mu}\phi=\partial_{\mu}\phi \tag{2.2}$$

（2）For the tetrad vector $A_{(\alpha)},A^{(\alpha)}$：

$$\begin{aligned}
&\tilde{D}_{\mu}A_{(\alpha)}=\partial_{\mu}A_{(\alpha)}-\hat{\tilde{\Gamma}}^{(\beta)}_{(\alpha)\mu}A_{(\beta)}\\
&\tilde{D}_{\mu}A^{(\alpha)}=\partial_{\mu}A^{(\alpha)}+\hat{\tilde{\Gamma}}^{(\alpha)}_{(\beta)\mu}A^{(\beta)}
\end{aligned} \tag{2.3}$$

（3）For the coordinate vector A_{μ},A^{μ}：

$$\tilde{D}_\nu A_\mu = \partial_\nu A_\mu - \tilde{\Gamma}^\rho_{\mu\nu} A_\rho$$
$$\tilde{D}_\nu A^\mu = \partial_\nu A^\mu + \tilde{\Gamma}^\mu_{\rho\nu} A^\rho$$

......(2.4)

（4）For the spinor $\psi, \bar{\psi}$：

$$\tilde{D}_\nu \psi = \partial_\nu \psi + \hat{\tilde{\Gamma}}_\mu \psi$$
$$\tilde{D}_\nu \bar{\psi} = \partial_\nu \bar{\psi} - \bar{\psi}\hat{\tilde{\Gamma}}_\mu$$

......(2.5)

And there are the following relationships：

$$\tilde{\Gamma}^\rho_{\sigma\mu} = \frac{1}{2}\tilde{g}^{\rho\lambda}\left(\partial_\sigma \tilde{g}_{\lambda\mu} + \partial_\mu \tilde{g}_{\lambda\sigma} - \partial_\lambda \tilde{g}_{\mu\sigma}\right)$$
$$\hat{\tilde{\Gamma}}_\mu = -\frac{i}{4}\sigma_{(\alpha\beta)}\hat{\tilde{\Gamma}}^{(\alpha)}_{(\beta)\mu}, \sigma_{(\alpha\beta)} = \frac{i}{2}\left(\gamma_{(\alpha)}\gamma_{(\beta)} - \gamma_{(\beta)}\gamma_{(\alpha)}\right)$$
$$\hat{\tilde{\Gamma}}^\alpha_{(\beta)\mu} = -\tilde{\lambda}^\nu_{(\beta)}\partial_\mu \tilde{\lambda}^{(\alpha)}_\nu + \tilde{\Gamma}^\rho_{\nu\mu}\tilde{\lambda}^{(\alpha)}_\rho \tilde{\lambda}^\nu_{(\beta)}$$
$$\tilde{D}_\mu \tilde{\lambda}^{(\alpha)}_\nu = 0, \tilde{D}_\lambda \tilde{g}_{\mu\nu} = 0, \tilde{D}_\lambda \tilde{g}^{\mu\nu} = 0, \tilde{D}_\mu \gamma_{(\alpha)} = \partial_\mu \gamma_{(\alpha)} = 0$$
$$\hat{\tilde{\Gamma}}_{(\alpha\beta)\mu} = \eta_{(\alpha\gamma)}\hat{\tilde{\Gamma}}^{(\gamma)}_{(\beta)\mu}, \hat{\tilde{\Gamma}}^{(\alpha\beta)}_\mu = \eta^{(\alpha\gamma)}\hat{\tilde{\Gamma}}^{(\beta)}_{(\gamma)\mu}$$
$$\tilde{\Gamma}^\rho_{\mu\nu} = \tilde{\Gamma}^\rho_{\nu\mu}, \hat{\tilde{\Gamma}}_{(\alpha\beta)\mu} = -\hat{\tilde{\Gamma}}_{(\beta\alpha)\mu}, \hat{\tilde{\Gamma}}^{(\alpha\beta)}_\mu = -\hat{\tilde{\Gamma}}^{(\beta\alpha)}_\mu$$

(2.6)

Let $\tilde{R}^\rho_{\sigma\mu\nu}, \hat{\tilde{R}}^{(\alpha)}_{(\beta)\mu\nu}, \hat{\tilde{R}}_{\mu\nu}$ are the coordinate, tetrad and spinor curvature tensors of a given Riemannian spacetime, there are:

$$\tilde{R}^\rho_{\sigma\mu\nu} = \partial_\mu \tilde{\Gamma}^\rho_{\sigma\nu} - \partial_\nu \tilde{\Gamma}^\rho_{\sigma\mu} + \tilde{\Gamma}^\rho_{\delta\mu}\tilde{\Gamma}^\delta_{\sigma\nu} - \tilde{\Gamma}^\rho_{\delta\nu}\tilde{\Gamma}^\delta_{\sigma\mu}$$
$$\hat{\tilde{R}}^{(\alpha)}_{(\beta)\mu\nu} = \partial_\mu \hat{\tilde{\Gamma}}^{(\alpha)}_{(\beta)\nu} - \partial_\nu \hat{\tilde{\Gamma}}^{(\alpha)}_{(\beta)\mu} + \hat{\tilde{\Gamma}}^{(\alpha)}_{(\gamma)\mu}\hat{\tilde{\Gamma}}^{(\gamma)}_{(\beta)\nu} - \hat{\tilde{\Gamma}}^{(\alpha)}_{(\gamma)\nu}\hat{\tilde{\Gamma}}^{(\gamma)}_{(\beta)\mu}$$
$$\hat{\tilde{R}}_{\mu\nu} = \partial_\mu \hat{\tilde{\Gamma}}_\nu - \partial_\nu \hat{\tilde{\Gamma}}_\mu + \hat{\tilde{\Gamma}}_\mu \hat{\tilde{\Gamma}}_\nu - \hat{\tilde{\Gamma}}_\nu \hat{\tilde{\Gamma}}_\mu$$
$$\hat{\tilde{R}}^\rho_{\sigma\mu\nu} = \tilde{\lambda}^\rho_{(\alpha)}\tilde{\lambda}^{(\beta)}_\sigma \hat{\tilde{R}}^{(\alpha)}_{(\beta)\mu\nu}, \tilde{R}^\rho_{\sigma\mu\nu} = \hat{\tilde{R}}^\rho_{\sigma\mu\nu}, \hat{\tilde{R}}_{\mu\nu} = -\frac{i}{4}\sigma^{(\alpha\beta)}\hat{\tilde{R}}_{(\alpha\beta)\mu\nu}$$

(2.7)

For the scalar ϕ 、 vector $A_{(\alpha)}, A^{(\alpha)}, A_\mu, A^\mu$ and spinor $\psi, \bar{\psi}$,there are：

$$\tilde{D}_\mu \tilde{D}_\nu \phi - \tilde{D}_\nu \tilde{D}_\mu \phi = 0$$
$$\tilde{D}_\mu \tilde{D}_\nu A_{(\alpha)} - \tilde{D}_\nu \tilde{D}_\mu A_{(\alpha)} = -\hat{\tilde{R}}^{(\beta)}_{(\alpha)\mu\nu} A_{(\beta)}$$
$$\tilde{D}_\mu \tilde{D}_\nu A^{(\alpha)} - \tilde{D}_\nu \tilde{D}_\mu A^{(\alpha)} = \hat{\tilde{R}}^{(\alpha)}_{(\beta)\mu\nu} A^{(\beta)}$$
$$\tilde{D}_\mu \tilde{D}_\nu A_\rho - \tilde{D}_\nu \tilde{D}_\mu A_\rho = -\tilde{R}^\sigma_{\rho\mu\nu} A_\sigma$$
$$\tilde{D}_\mu \tilde{D}_\nu A^\rho - \tilde{D}_\nu \tilde{D}_\mu A^\rho = \tilde{R}^\rho_{\sigma\mu\nu} A^\sigma$$
$$\tilde{D}_\mu \tilde{D}_\nu \psi - \tilde{D}_\nu \tilde{D}_\mu \psi = \hat{\tilde{R}}_{\mu\nu}\psi$$
$$\tilde{D}_\mu \tilde{D}_\nu \bar{\psi} - \tilde{D}_\nu \tilde{D}_\mu \bar{\psi} = -\bar{\psi}\hat{\tilde{R}}_{\mu\nu}$$

(2.8)

Geometry of Curved Spacetime with Torsion

Let $\lambda^{(\alpha)}_\mu, \lambda^\mu_{(\alpha)}$ is the tetrad field in the curved spacetime with torsion M^4 , $g_{\mu\nu,}g^{\mu\nu}$ is the coordinate metric tensor in the curved spacetime with torsion M^4, $\eta_{(\alpha\beta)}, \eta^{(\alpha\beta)}$ is the tetrad metric tensor in the curved spacetime with torsion M^4,

$\delta^{\nu}_{\mu},\delta^{(\alpha)}_{(\beta)}$ is the unit coordinate tensor and the unit tetrad tensor in the curved spacetime with torsion M^4. In the curved spacetime with torsion M^4, Coordinate indicators use $g_{\mu\nu},g^{\mu\nu}$ to rise and fall ,tetrad indicators use $\eta_{(\alpha\beta)},\eta^{(\alpha\beta)}$ to rise and fall, Using $\lambda^{(\alpha)}_{\mu},\lambda^{\mu}_{(\alpha)}$ as the transformation between tetrad indicators and coordinate indicators, there are the following relations:

$$
\begin{aligned}
&\lambda^{(\alpha)}_{\mu}\lambda^{\nu}_{(\alpha)}=\delta^{\nu}_{\mu},\lambda^{(\alpha)}_{\rho}\lambda^{\rho}_{(\beta)}=\delta^{(\alpha)}_{(\beta)}\\
&\lambda_{(\alpha)\mu}=\eta_{(\alpha\beta)}\lambda^{(\beta)}_{\mu}=g_{\mu\nu}\lambda^{\nu}_{(\alpha)}\\
&\lambda^{(\alpha)\mu}=\eta^{(\alpha\beta)}\lambda^{\mu}_{(\beta)}=g^{\mu\nu}\lambda^{(\alpha)}_{\nu}\\
&\eta_{(\alpha\beta)}=\lambda^{\mu}_{(\alpha)}\lambda_{(\beta)\mu},\eta^{(\alpha\beta)}=\lambda^{(\alpha)\mu}\lambda^{(\beta)}_{\mu}\\
&g_{\mu\nu}=\lambda^{(\alpha)}_{\mu}\lambda_{(\alpha)\nu},g^{\mu\nu}=\lambda^{(\alpha)\mu}\lambda^{\nu}_{(\alpha)}
\end{aligned}
\tag{2.9}
$$

where:

$$
\eta_{(\alpha\beta)}=\begin{pmatrix}1 & 0\\ 0 & -\delta_{(ab)}\end{pmatrix},\eta^{(\alpha\beta)}=\begin{pmatrix}1 & 0\\ 0 & -\delta^{(ab)}\end{pmatrix}
$$

$$\alpha,\beta=0,1,2,3;a,b=1,2,3$$

Let $\Gamma^{\rho}_{\sigma\mu},\hat{\Gamma}^{(\alpha)}_{(\beta)\mu},\hat{\Gamma}_{\mu}$ are the basic coordinate affine connection (Christoffel symbol), the basic frame affine connection and the spinor connection in a curved spacetime with torsion, we can define the basic covariant derivative D_{μ},D^{μ} in curved spacetime with torsion:

（1）For the scalar ϕ：

$$
D_{\mu}\phi=\partial_{\mu}\phi \tag{2.10}
$$

（2）For the metard vector $A_{(\alpha)},A^{(\alpha)}$：

$$
\begin{aligned}
&D_{\mu}A_{(\alpha)}=\partial_{\mu}A_{(\alpha)}-\hat{\Gamma}^{(\beta)}_{(\alpha)\mu}A_{(\beta)}\\
&D_{\mu}A^{(\alpha)}=\partial_{\mu}A^{(\alpha)}+\hat{\Gamma}^{(\alpha)}_{(\beta)\mu}A^{(\beta)}
\end{aligned}
\tag{2.11}
$$

（3）For the coordinate vector A_{μ},A^{μ}：

$$
\begin{aligned}
&D_{\nu}A_{\mu}=\partial_{\nu}A_{\mu}-\Gamma^{\rho}_{\mu\nu}A_{\rho}\\
&D_{\nu}A^{\mu}=\partial_{\nu}A^{\mu}+\Gamma^{\mu}_{\rho\nu}A^{\rho}
\end{aligned}
$$

......(2.12)

（4）For the spinor $\psi,\bar{\psi}$：

$$
\begin{aligned}
&D_{\nu}\psi=\partial_{\nu}\psi+\hat{\Gamma}_{\mu}\psi\\
&D_{\nu}\bar{\psi}=\partial_{\nu}\bar{\psi}-\bar{\psi}\hat{\Gamma}_{\mu}
\end{aligned}
$$

......(2.13)

Let $F_{\lambda\mu\nu},A_{\mu}$ are the torsion tensor and the electromagnetic potential in the curved spacetime with torsion,there are the following relationships:

$$
\begin{aligned}
&\Gamma^{\rho}_{\sigma\mu}=\frac{1}{2}g^{\rho\lambda}\left(\partial_{\sigma}g_{\lambda\mu}+\partial_{\mu}g_{\lambda\sigma}-\partial_{\lambda}g_{\mu\sigma}\right)\\
&F_{\lambda\mu\nu}=\lambda_{(\alpha)\lambda}\left(D_{\mu}\lambda^{(\alpha)}_{\nu}-D_{\nu}\lambda^{(\alpha)}_{\mu}\right),F_{\mu}=g^{\rho\sigma}F_{\rho\sigma\mu}\\
&H_{\lambda\mu\nu}=F_{\lambda\mu\nu}+F_{\mu\lambda\nu}-F_{\nu\lambda\mu},H_{\mu}=g^{\rho\sigma}H_{\rho\sigma\mu}\\
&\hat{\Gamma}_{\mu}=-\frac{i}{4}\sigma_{(\alpha\beta)}\hat{\Gamma}^{(\alpha)}_{(\beta)\mu}-i\frac{e}{\hbar c}A_{\mu}\\
&\sigma_{(\alpha\beta)}=\frac{i}{2}\left(\gamma_{(\alpha)}\gamma_{(\beta)}-\gamma_{(\beta)}\gamma_{(\alpha)}\right),D_{\mu}\lambda^{(\alpha)}_{\nu}=-\frac{1}{2}H_{\mu\nu}{}^{(\alpha)}\neq 0\\
&D_{\lambda}g_{\mu\nu}=0,D_{\lambda}g^{\mu\nu}=0,D_{\mu}\gamma_{(\alpha)}=\partial_{\mu}\gamma_{(\alpha)}=0\\
&\hat{\Gamma}_{(\alpha\beta)\mu}=\eta_{(\alpha\gamma)}\hat{\Gamma}^{(\gamma)}_{(\beta)\mu},\hat{\Gamma}^{(\alpha\beta)}_{\mu}=\eta^{(\alpha\gamma)}\hat{\Gamma}^{(\beta)}_{(\gamma)\mu}\\
&\Gamma^{\rho}_{\mu\nu}=\Gamma^{\rho}_{\nu\mu},\hat{\Gamma}_{(\alpha\beta)\mu}=-\hat{\Gamma}_{(\beta\alpha)\mu},\hat{\Gamma}^{(\alpha\beta)}_{\mu}=-\hat{\Gamma}^{(\beta\alpha)}_{\mu}
\end{aligned}
$$

(2.14)

Let $R^{\rho}_{\sigma\mu\nu},\hat{R}^{(\alpha)}_{(\beta)\mu\nu},\hat{R}_{\mu\nu},F_{\mu\nu}$ are the coordinate curvature tensor, the frame curvature tensor, the spinor curvature tensor and the electromagnetic field tensor of a curved spacetime with torsion, there are:

$$
\begin{aligned}
&R^{\rho}_{\sigma\mu\nu}=\partial_{\mu}\Gamma^{\rho}_{\sigma\nu}-\partial_{\nu}\Gamma^{\rho}_{\sigma\mu}+\Gamma^{\rho}_{\delta\mu}\Gamma^{\delta}_{\sigma\nu}-\Gamma^{\rho}_{\delta\nu}\Gamma^{\delta}_{\sigma\mu}\\
&\hat{R}^{(\alpha)}_{(\beta)\mu\nu}=\partial_{\mu}\hat{\Gamma}^{(\alpha)}_{(\beta)\nu}-\partial_{\nu}\hat{\Gamma}^{(\alpha)}_{(\beta)\mu}+\hat{\Gamma}^{(\alpha)}_{(\gamma)\mu}\hat{\Gamma}^{(\gamma)}_{(\beta)\nu}-\hat{\Gamma}^{(\alpha)}_{(\gamma)\nu}\hat{\Gamma}^{(\gamma)}_{(\beta)\mu}\\
&\hat{R}_{\mu\nu}=\partial_{\mu}\hat{\Gamma}_{\nu}-\partial_{\nu}\hat{\Gamma}_{\mu}+\hat{\Gamma}_{\mu}\hat{\Gamma}_{\nu}-\hat{\Gamma}_{\nu}\hat{\Gamma}_{\mu}\\
&F_{\mu\nu}=D_{\mu}A_{\nu}-D_{\nu}A_{\mu}\\
&\hat{R}^{\rho}_{\sigma\mu\nu}=\lambda^{\rho}_{(\alpha)}\lambda^{(\beta)}_{\sigma}\hat{R}^{(\alpha)}_{(\beta)\mu\nu},\hat{R}_{\mu\nu}=-\frac{i}{4}\sigma^{(\alpha\beta)}\hat{R}_{(\alpha\beta)\mu\nu}-i\frac{e}{\hbar c}F_{\mu\nu}\\
&R^{\rho}_{\sigma\mu\nu}=\hat{R}^{\rho}_{\sigma\mu\nu}+\frac{1}{2}\left(D_{\mu}H_{\nu\sigma}{}^{\rho}-D_{\nu}H_{\mu\sigma}{}^{\rho}\right)-\frac{1}{4}\left(H_{\mu m}{}^{\rho}H_{\nu\sigma}{}^{m}-H_{\nu m}{}^{\rho}H_{\mu\sigma}{}^{m}\right)
\end{aligned}
$$

(2.15)

For the scalar ϕ 、 vector $A_{(\alpha)},A^{(\alpha)},A_{\mu},A^{\mu}$ andspinor $\psi,\bar{\psi}$,there are：

$$
\begin{aligned}
&D_{\mu}D_{\nu}\phi-D_{\nu}D_{\mu}\phi=0\\
&D_{\mu}D_{\nu}A_{(\alpha)}-D_{\nu}D_{\mu}A_{(\alpha)}=-\hat{R}^{(\beta)}_{(\alpha)\mu\nu}A_{(\beta)}\\
&D_{\mu}D_{\nu}A^{(\alpha)}-D_{\nu}D_{\mu}A^{(\alpha)}=\hat{R}^{(\alpha)}_{(\beta)\mu\nu}A^{(\beta)}\\
&D_{\mu}D_{\nu}A_{\rho}-D_{\nu}D_{\mu}A_{\rho}=-R^{\sigma}_{\rho\mu\nu}A_{\sigma}\\
&D_{\mu}D_{\nu}A^{\rho}-D_{\nu}D_{\mu}A^{\rho}=R^{\rho}_{\sigma\mu\nu}A^{\sigma}\\
&D_{\mu}D_{\nu}\psi-D_{\nu}D_{\mu}\psi=\hat{R}_{\mu\nu}\psi\\
&D_{\mu}D_{\nu}\bar{\psi}-D_{\nu}D_{\mu}\bar{\psi}=-\bar{\psi}\hat{R}_{\mu\nu}
\end{aligned}
$$

(2.16)

Formulation of Curved Spacetime with Torsion in Riemannian Background Spacetime

The deviation of a curved spacetime M^4 with torsion from a specified Riemannian background spacetime $\tilde{M}^4$ is mainly divided into two types: the curved deviation and the twisted deviation. The curved deviation is described by the difference between the tetrad field $\lambda^{(\alpha)}_{\mu}$ in the curved spacetime M^4 with torsion and the tetrad field $\tilde{\lambda}^{(\alpha)}_{\mu}$ in the specified Riemannian background spacetime $\tilde{M}^4$. The twisted deviation is described by the difference physical quantity between the tetrad affine connection $\hat{\Gamma}^{(\alpha)}_{(\beta)\mu}$ in the curved spacetime M^4 with torsion and the tetrad affine connection $\tilde{\hat{\Gamma}}^{(\alpha)}_{(\beta)\mu}$ in the specified Riemannian background spacetime $\tilde{M}^4$.

Formulation of Curved Deviation Tensor and Coordinate Affine Connection in Riemannian Background Spacetime

We can construct the curved deviation tensor $\varpi_\mu{}^\nu, \omega_\nu{}^\mu$ from the tetrad field $\lambda_\mu^{(\alpha)}, \lambda_{(\alpha)}^\nu$ in the curved spacetime M^4 with torsion and the tetrad field $\tilde{\lambda}_\mu^{(\alpha)}, \tilde{\lambda}_{(\alpha)}^\nu$ in the specified Riemannian background spacetime $\tilde{M}^4$:

$$\lambda_\mu^{(\alpha)} = \omega_\mu{}^\nu \tilde{\lambda}_\nu^{(\alpha)}, \lambda_{(\alpha)}^\mu = \tilde{\lambda}_{(\alpha)}^\nu \varpi_\nu{}^\mu$$

......(3.1)

So, there are:

$$\lambda_\mu^{(\alpha)} \tilde{\lambda}_{(\alpha)}^\nu = \omega_\mu{}^\nu, \lambda_{(\alpha)}^\mu \tilde{\lambda}_\nu^{(\alpha)} = \varpi_\nu{}^\mu,$$
$$g_{\mu\nu} = \lambda_\mu^{(\alpha)} \eta_{(\alpha\beta)} \lambda_\nu^{(\beta)} = \omega_\mu{}^\rho \omega_\mu{}^\sigma \tilde{g}_{\rho\sigma}$$
$$g^{\mu\nu} = \lambda_{(\alpha)}^\mu \eta^{(\alpha\beta)} \lambda_{(\beta)}^\nu = \tilde{g}^{\rho\sigma} \varpi_\rho{}^\mu \varpi_\sigma{}^\nu$$

......(3.2)

This relation decomposes the tetrad field $\lambda_\mu^{(\alpha)}, \lambda_{(\alpha)}^\mu$ in a curved spacetime M^4 with torsion into the tetrad field $\tilde{\lambda}_\mu^{(\alpha)}, \tilde{\lambda}_{(\alpha)}^\mu$ of a given Riemannian background spacetime and the curved deviation tensor $\omega_\mu{}^\nu, \varpi_\nu{}^\mu$.The tetrad field $\tilde{\lambda}_\mu^{(\alpha)}, \tilde{\lambda}_{(\alpha)}^\mu$ represent that effects of the geometry of the Riemannian background spacetime. The curved deviation tensor $\omega_\mu{}^\nu, \varpi_\nu{}^\mu$ represents the physical effect of the gravitational field.

From the properties of the tetrad field $\lambda_\mu^{(\alpha)}, \tilde{\lambda}_\mu^{(\alpha)}$, we can prove the following relation:

$$\varpi_\mu{}^\rho \omega_\rho{}^\nu = \varpi_\rho{}^\nu \omega_\mu{}^\rho = \delta_\mu^\nu$$

...(3.3)

Let the coordinate affine connection (Christoffel symbol) in a curved spacetime M^4 is $\Gamma_{\sigma\mu}^\rho$,there are:

$$\begin{aligned}
\Gamma_{\mu\nu}^\rho &= \frac{1}{2} g^{\rho\lambda} \left(\partial_\mu g_{\lambda\nu} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu} \right) \\
&= \frac{1}{2} g^{\rho\lambda} \begin{pmatrix} \left(\tilde{D}_\mu g_{\lambda\nu} + \tilde{\Gamma}_{\lambda\mu}^\delta g_{\delta\nu} + \tilde{\Gamma}_{\nu\mu}^\delta g_{\lambda\delta} \right) \\ + \left(\tilde{D}_\nu g_{\lambda\mu} + \tilde{\Gamma}_{\lambda\nu}^\delta g_{\delta\mu} + \tilde{\Gamma}_{\mu\nu}^\delta g_{\lambda\delta} \right) \\ - \left(\tilde{D}_\lambda g_{\mu\nu} + + \tilde{\Gamma}_{\mu\lambda}^\delta g_{\delta\nu} + \tilde{\Gamma}_{\nu\lambda}^\delta g_{\mu\delta} \right) \end{pmatrix} \\
&= \tilde{\Gamma}_{\mu\nu}^\rho + \frac{1}{2} g^{\rho\lambda} \left(\tilde{D}_\mu g_{\lambda\nu} + \tilde{D}_\nu g_{\lambda\mu} - \tilde{D}_\lambda g_{\mu\nu} \right)
\end{aligned}$$

......(3.4)

Substituting (3.2) into the above equation, we get:

$$\Gamma_{\mu\nu}^\rho = \tilde{\Gamma}_{\mu\nu}^\rho + \varpi_m{}^\rho \tilde{D}_\nu \omega_\mu{}^m + \frac{1}{2} \varpi_m{}^\rho \varpi^{m\lambda} \hat{H}_{\nu\mu\lambda}$$

$$\begin{Bmatrix} \hat{H}_{\lambda\mu\nu} = \hat{F}_{\lambda\mu\nu} + \hat{F}_{\mu\lambda\nu} - \hat{F}_{\nu\lambda\mu} \\ \hat{F}_{\lambda\mu\nu} = \omega_{\lambda l} \left(\tilde{D}_\mu \omega_\nu{}^l - \tilde{D}_\nu \omega_\mu{}^l \right) \\ \tilde{F}_{\mu\nu}^l = \tilde{D}_\mu \omega_\nu{}^l - \tilde{D}_\nu \omega_\mu{}^l \end{Bmatrix}$$

(3.5)

Therefore, $\Gamma_{\mu\nu}^\rho$ can be written as:

$$\Gamma^{\rho}_{\mu\nu} = \tilde{\Gamma}^{\rho}_{\mu\nu} + N^{\rho}_{\mu\nu}$$

......(3.6)

Where, $N^{\rho}_{\mu\nu} = \frac{1}{2} g^{\rho\lambda} \left(\tilde{D}_{\mu} g_{\lambda\nu} + \tilde{D}_{\nu} g_{\lambda\mu} - \tilde{D}_{\lambda} g_{\mu\nu} \right)$ is a tensor in curved spacetime with torsion, called the coordinate affine connection deviation tensor.

$$\begin{aligned} N_{\mu} = N^{\rho}_{\mu\rho} &= \frac{1}{2} g^{\rho\lambda} \tilde{D}_{\mu} g_{\rho\lambda} \\ &= \frac{1}{2} \tilde{g}^{nm} \varpi_{n}{}^{\rho} \varpi_{m}{}^{\lambda} \tilde{D}_{\mu} \left(\tilde{g}_{kl} \omega_{\rho}{}^{k} \omega_{\lambda}{}^{l} \right) \\ &= \frac{1}{2} \tilde{g}^{nm} \varpi_{n}{}^{\rho} \varpi_{m}{}^{\lambda} \left(\tilde{g}_{kl} \tilde{D}_{\mu} \omega_{\rho}{}^{k} \omega_{\lambda}{}^{l} + \tilde{g}_{kl} \omega_{\rho}{}^{k} \tilde{D}_{\mu} \omega_{\lambda}{}^{l} \right) \\ &= \varpi_{k}{}^{\rho} \tilde{D}_{\mu} \omega_{\rho}{}^{k} \end{aligned}$$

Formulation of Twisted Deviation Tensor and Frame Affine Connection in Riemannian Background Spacetime

Using the tetrad affine connection $\hat{\Gamma}^{(\alpha)}_{(\beta)\mu}$ in curved spacetime M^4 with torsion and the tetrad affine connection $\hat{\tilde{\Gamma}}^{(\alpha)}_{(\beta)\mu}$ in Riemannian background spacetime $\tilde{M}^4$, we can construct the twisted deviation tensor $M^{(\alpha)}_{(\beta)\mu}$ in the following formula:

$$\hat{\Gamma}^{(\alpha)}_{(\beta)\mu} = \hat{\tilde{\Gamma}}^{(\alpha)}_{(\beta)\mu} + M^{(\alpha)}_{(\beta)\mu}$$

......(3.7)

It can be proved that $M^{(\alpha)}_{(\beta)\mu}$ is a tensor, which describes the degree of distortion of the Riemannian background spacetime $\tilde{M}^4$ by the curved spacetime M^4 with torsion, and is called the distortion tensor. Let $M^{(\alpha)}_{(\beta)\mu} = \tilde{\lambda}^{(\alpha)}_{\rho} \tilde{\lambda}^{\sigma}_{(\beta)} M^{\rho}_{\sigma\mu}$, there is:

$$\widehat{M}^{\rho}_{\sigma\mu} = \lambda^{\rho}_{(\alpha)} \lambda^{(\beta)}_{\sigma} M^{(\alpha)}_{(\beta)\mu} = \varpi_{n}{}^{\rho} \omega_{\sigma}{}^{m} M^{n}_{m\mu}$$

The Relation between the Background Covariant Differential $\tilde{D}_{\mu}, \tilde{D}^{\mu}$ and the Basic Covariant Differential D_{μ}, D^{μ}

According to the above definitions of various connections, we obtain the following covariant differential relations:

(1) For the scalar ϕ:

$$D_{\mu}\phi = \tilde{D}_{\mu}\phi = \partial_{\mu}\phi$$

......(3.8)

(2) For the tetrad vector $A_{(\alpha)}, A^{(\alpha)}$:

$$\begin{aligned} D_{\mu} A_{(\alpha)} &= \tilde{D}_{\mu} A_{(\alpha)} - M^{(\beta)}_{(\alpha)\mu} A_{(\beta)} \\ D_{\mu} A^{(\alpha)} &= \tilde{D}_{\mu} A^{(\alpha)} + M^{(\alpha)}_{(\beta)\mu} A^{(\beta)} \end{aligned}$$

......(3.9)

(3) For the coordinate vector A_{μ}, A^{μ}:

$$\begin{aligned} D_{\mu} A_{\nu} &= \tilde{D}_{\mu} A_{\nu} - N^{\rho}_{\nu\mu} A_{\rho} \\ D_{\mu} A^{\nu} &= \tilde{D}_{\mu} A^{\nu} + N^{\nu}_{\rho\mu} A^{\rho} \end{aligned}$$

......(3.10)

（4）For the spinor $\psi,\bar{\psi}$：

$$D_\mu \psi = \tilde{D}_\mu \psi + \Xi_\mu \psi, D_\mu \bar{\psi} = \tilde{D}_\mu \bar{\psi} - \bar{\psi}\Xi_\mu$$
$$\Xi_\mu = -\frac{i}{4}\sigma^{(\alpha\beta)} M_{(\alpha\beta)\mu} - i\frac{e}{\hbar c}A_\mu$$

......(3.11)

The Background Formulation of the Torsion Tensor in the Curved Spacetime with Torsion

The torsion tensor (the field strength tensor of the gravitational field) in the torsion curved spacetime M^4 is $F_{\lambda\mu\nu}$, so there are:

$$\begin{aligned} F_{\lambda\mu\nu} &= \lambda_{(\alpha)\lambda}\left(D_\mu \lambda_\nu^{(\alpha)} - D_\nu \lambda_\mu^{(\alpha)}\right) \\ &= \lambda_{(\alpha)\lambda}\left(\tilde{D}_\mu \lambda_\nu^{(\alpha)} - \tilde{D}_\nu \lambda_\mu^{(\alpha)} + M_{(\beta)\mu}^{(\alpha)} \lambda_\nu^{(\beta)} - M_{(\beta)\nu}^{(\alpha)} \lambda_\mu^{(\beta)}\right) \\ &= \widehat{F}_{\lambda\mu\nu} + \widehat{M}_{\lambda\nu\mu} - \widehat{M}_{\lambda\mu\nu} \\ &= \widehat{F}_{\lambda\mu\nu} + \widehat{M}_{\mu\lambda\nu} - \widehat{M}_{\nu\lambda\mu} \end{aligned}$$
$$F_\mu = \widehat{F}_\mu - \widehat{M}_\mu, \widehat{M}_\mu = g^{\rho\sigma}\widehat{M}_{\mu\rho\sigma}$$

......(3.12)

Substitute the above relation into $H_{\lambda\mu\nu} = F_{\lambda\mu\nu} + F_{\mu\lambda\nu} - F_{\nu\lambda\mu}$,we get：

$$H_{\lambda\mu\nu} = \widehat{H}_{\lambda\mu\nu} + 2\widehat{M}_{\mu\nu\lambda}, H_\lambda = \widehat{H}_\lambda - 2\widehat{M}_\lambda$$

......(3.13)

Where $\widehat{H}_{\lambda\mu\nu} = \widehat{F}_{\lambda\mu\nu} + \widehat{F}_{\mu\lambda\nu} - \widehat{F}_{\nu\lambda\mu}, \widehat{F}_{\lambda\mu\nu} = \omega_{\lambda n}\tilde{F}^n{}_{\mu\nu}$ 。

In the ECT spin-gravity theory, when $\beta_1 = -\beta_2 = 1, \beta_3 = 0, \alpha'' = 0, \gamma = 0$ has:：

$$\begin{aligned} K_{\lambda\mu\nu} &= \left(F_{\lambda\mu\nu} - F_{\mu\lambda\nu} + F_{\nu\lambda\mu}\right) \\ &= \begin{pmatrix} \widehat{F}_{\lambda\mu\nu} + \widehat{M}_{\mu\lambda\nu} - \widehat{M}_{\nu\lambda\mu} \\ -\left(\widehat{F}_{\mu\lambda\nu} + \widehat{M}_{\lambda\mu\nu} - \widehat{M}_{\nu\mu\lambda}\right) \\ +\left(\widehat{F}_{\nu\lambda\mu} + \widehat{M}_{\lambda\nu\mu} - \widehat{M}_{\mu\nu\lambda}\right) \end{pmatrix} \\ &= \begin{pmatrix} \widehat{F}_{\lambda\mu\nu} - \widehat{F}_{\mu\lambda\nu} + \widehat{F}_{\nu\lambda\mu} \\ +2\widehat{M}_{\mu\lambda\nu} - 2\widehat{M}_{\nu\lambda\mu} + 2\widehat{M}_{\nu\mu\lambda} \end{pmatrix} \\ &= \widehat{K}_{\lambda\mu\nu} - 2\left(\widehat{M}_{\mu\nu\lambda} - \left(\widehat{M}_{\mu\lambda\nu} - \widehat{M}_{\nu\lambda\mu}\right)\right) \end{aligned}$$

......(3.14)

Where $\widehat{K}_{\lambda\mu\nu} = \widehat{F}_{\lambda\mu\nu} - \widehat{F}_{\mu\lambda\nu} + \widehat{F}_{\nu\lambda\mu}$.

The Background Formulation of the Curvature Tensor and the Electromagnetic Field Tensor in the Curved Spacetime with Torsion

Let $\hat{R}_{(\beta)\mu\nu}^{(\alpha)}, \hat{\hat{R}}_{(\beta)\mu\nu}^{(\alpha)}$ be the tetrad curvature tensor in a curved spacetime with torsion and in a Riemannian background spacetime, respectively. Substituting $\hat{\Gamma}_{(\beta)\mu}^{(\alpha)} = \hat{\hat{\Gamma}}_{(\beta)\mu}^{(\alpha)} + M_{(\beta)\mu}^{(\alpha)}$ in the definition of $\hat{R}_{(\beta)\mu\nu}^{(\alpha)}$, we get：

$$\hat{R}^{(\alpha)}_{(\beta)\mu\nu} = \hat{\tilde{R}}^{(\alpha)}_{(\beta)\mu\nu} + \hat{\hat{R}}^{(\alpha)}_{(\beta)\mu\nu}$$

$$\left\{\begin{aligned} \hat{\hat{R}}^{(\alpha)}_{(\beta)\mu\nu} &= \hat{D}_\mu M^{(\alpha)}_{(\beta)\nu} - \hat{D}_\nu M^{(\alpha)}_{(\beta)\mu} + M^{(\alpha)}_{(\gamma)\mu} M^{(\gamma)}_{(\beta)\nu} - M^{(\alpha)}_{(\gamma)\nu} M^{(\gamma)}_{(\beta)\mu} \\ &= \tilde{D}_\mu M^{(\alpha)}_{(\beta)\nu} - \tilde{D}_\nu M^{(\alpha)}_{(\beta)\mu} + M^{(\alpha)}_{(\gamma)\mu} M^{(\gamma)}_{(\beta)\nu} - M^{(\alpha)}_{(\gamma)\nu} M^{(\gamma)}_{(\beta)\mu} \\ &= \tilde{\lambda}^{(\alpha)}_\rho \tilde{\lambda}^\sigma_{(\beta)} \left(\tilde{D}_\mu M^\rho_{\sigma\nu} - \tilde{D}_\nu M^\rho_{\sigma\mu} + M^\rho_{m\mu} M^m_{\sigma\nu} - M^\rho_{m\nu} M^m_{\sigma\mu} \right) \\ &= \tilde{\lambda}^{(\alpha)}_\rho \tilde{\lambda}^\sigma_{(\beta)} \hat{\hat{R}}^\rho_{\sigma\mu\nu}, \hat{\hat{R}}^\rho_{\sigma\mu\nu} = \tilde{D}_\mu M^\rho_{\sigma\nu} - \tilde{D}_\nu M^\rho_{\sigma\mu} + M^\rho_{m\mu} M^m_{\sigma\nu} - M^\rho_{m\nu} M^m_{\sigma\mu} \end{aligned}\right\}$$

……(3.15)

Where $\hat{\hat{R}}^{(\alpha)}_{(\beta)\mu\nu}$ is the effective field strength tensor of the spin field.

Let $R^\rho_{\sigma\mu\nu}, \tilde{R}^\rho_{\sigma\mu\nu}$ be the coordinate curvature tensor in a curved spacetime with torsion and in a Riemannian background spacetime, respectively. Substituting $\Gamma^\rho_{\mu\nu} = \tilde{\Gamma}^\rho_{\mu\nu} + N^\rho_{\mu\nu}$ in the definition of $R^\rho{}_{\sigma\mu\nu}$, we get:

$$R^\rho_{\sigma\mu\nu} = \tilde{R}^\rho_{\sigma\mu\nu} + \hat{R}^\rho_{\sigma\mu\nu}$$
$$\hat{R}^\rho_{\sigma\mu\nu} = \tilde{D}_\mu N^\rho_{\sigma\nu} - \tilde{D}_\nu N^\rho_{\sigma\mu} + N^\rho_{m\mu} N^m_{\sigma\nu} - N^\rho_{m\nu} N^m_{\sigma\mu}$$

……(3.16)

Let $\hat{\tilde{R}}^{(\alpha)}_{(\beta)\mu\nu}, \tilde{R}^\rho_{\sigma\mu\nu}$ be the frame curvature tensor and the coordinate curvature tensor in the Riemannian background spacetime, respectively. There is:

$$\tilde{D}_\mu \tilde{D}_\nu \lambda^{(\alpha)}_\sigma - \tilde{D}_\nu \tilde{D}_\mu \lambda^{(\alpha)}_\sigma = \hat{\tilde{R}}^{(\alpha)}_{(\beta)\mu\nu} \lambda^{(\beta)}_\sigma - \tilde{R}^\rho_{\sigma\mu\nu} \lambda^{(\alpha)}_\rho$$

Substituting $\tilde{D}_\sigma \tilde{\lambda}^{(\alpha)}_\mu = 0$ into the above equation, we obtain:

$$\tilde{R}^\rho_{\sigma\mu\nu} = \tilde{\lambda}^\rho_{(\alpha)} \hat{\tilde{R}}^{(\alpha)}_{(\beta)\mu\nu} \tilde{\lambda}^{(\beta)}_\sigma$$

……(3.17)

Let A_μ is the electromagnetic field potential vector and $F_{\mu\nu}$ is the electromagnetic field tensor, then we get:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = \tilde{D}_\mu A_\nu - \tilde{D}_\nu A_\mu$$

…(3.18)

Riemannian Spacetime Background Formulation of the Equations of Motion

The following is a background dependent formulation of the equations of motion for the ECT spin-gravity field theory with the Einstein limit

Dirac Electron Field Equation of Motion

Since there are: $D_\mu \psi = \tilde{D}_\mu \psi + \Xi_\mu \psi, D_\mu \bar{\psi} = \tilde{D}_\mu \bar{\psi} - \bar{\psi} \Xi_\mu$, according to the Dirac electron field equation of motion of the ECT spin-gravitational field theory:

$$\begin{cases} \frac{1}{2}\left(i\hbar \gamma^\mu D_\mu \psi + i\hbar D_\mu (\gamma^\mu \psi) \right) - mc\psi = 0 \\ \frac{1}{2}\left(i\hbar D_\mu \bar{\psi} \gamma^\mu + i\hbar D_\mu (\bar{\psi} \gamma^\mu) \right) + mc\bar{\psi} = 0 \end{cases}$$

We obtain the formulation of the Dirac electron field equation on the Riemannian background spacetime as:

$$\begin{cases} i\hbar \gamma^\mu \tilde{D}_\mu \psi + \frac{1}{2} i\hbar \tilde{D}_\mu \gamma^\mu \psi - \frac{1}{2}\hbar \left(\gamma^\mu \Xi_\mu + \Xi_\mu \gamma^\mu \right) \psi - mc\psi = 0 \\ i\hbar \tilde{D}_\mu \bar{\psi} \gamma^\mu + \frac{1}{2} i\hbar \bar{\psi} \tilde{D}_\mu \gamma^\mu + \frac{1}{2}\hbar \bar{\psi} \left(\gamma^\mu \Xi_\mu + \Xi_\mu \gamma^\mu \right) + mc\bar{\psi} = 0 \end{cases}$$

……(3.20)

Maxwell's Equation of Electromagnetic Field Motion

Since there is: $D_\nu F^{\mu\nu} = \tilde{D}_\nu F^{\mu\nu} + N_m F^{\mu m}$, according to Maxwell's electromagnetic field equation of motion:

$$D_\nu F^{\mu\nu} = 4\pi j_e^\mu$$

The equations of motion of Maxwell's electromagnetic field on a Riemannian background spacetime are expressed as:

$$\tilde{D}_\nu F^{\mu\nu} + N_m F^{\mu m} = 4\pi j_e^\mu$$

......(3.21)

Tang Fei Lin's spin field equation of motion

Because of:

$$\begin{aligned}
s_g^{(\alpha\beta)\mu} &= -\frac{c^4}{16\pi G}\beta\left(K^{(\alpha)\mu(\beta)} - K^{(\beta)\mu(\alpha)}\right) + \frac{c^4}{16\pi G}\alpha\left(Q_E{}^{(\alpha)\mu(\beta)} - Q_E{}^{(\beta)\mu(\alpha)}\right) \\
&= \frac{c^4}{8\pi G}\beta K^{\mu(\alpha\beta)} + \frac{c^4}{8\pi G}\alpha\left(F^{\mu(\alpha\beta)} - \lambda^{(\alpha)\mu}F^{(\beta)} + \lambda^{(\beta)\mu}F^{(\alpha)}\right) \\
&= \begin{pmatrix} \frac{c^4}{8\pi G}\beta\left(\hat{K}^{\mu(\alpha\beta)} - 2\left(\hat{M}^{(\alpha\beta)\mu} - \hat{M}^{(\alpha)\mu(\beta)} + \hat{M}^{(\beta)\mu(\alpha)}\right)\right) \\ + \frac{c^4}{8\pi G}\alpha\begin{pmatrix} \hat{F}^{\mu(\alpha\beta)} + \hat{M}^{(\alpha)\mu(\beta)} - \hat{M}^{(\beta)\mu(\alpha)} \\ -\lambda^{(\alpha)\mu}\left(\hat{F}^{(\beta)} - \hat{M}^{(\beta)}\right) + \lambda^{(\beta)\mu}\left(\hat{F}^{(\alpha)} - \hat{M}^{(\alpha)}\right) \end{pmatrix} \end{pmatrix} \\
&= \begin{pmatrix} \frac{c^4}{8\pi G}\left(\beta\hat{K}^{\mu(\alpha\beta)} + \alpha\left(\hat{F}^{\mu(\alpha\beta)} - \lambda^{(\alpha)\mu}\hat{F}^{(\beta)} + \lambda^{(\beta)\mu}\hat{F}^{(\alpha)}\right)\right) \\ -2\beta\frac{c^4}{8\pi G}\hat{M}^{(\alpha\beta)\mu} + \alpha\frac{c^4}{8\pi G}\left(\lambda^{(\alpha)\mu}\hat{M}^{(\beta)} - \lambda^{(\beta)\mu}\hat{M}^{(\alpha)}\right) \\ +(2\beta+\alpha)\frac{c^4}{8\pi G}\left(\hat{M}^{(\alpha)\mu(\beta)} - \hat{M}^{(\beta)\mu(\alpha)}\right) \end{pmatrix} \\
&= \frac{c^4}{8\pi G}\begin{pmatrix} \left(\beta\hat{K}^{\mu(\alpha\beta)} + \alpha\left(\hat{F}^{\mu(\alpha\beta)} - \left(\lambda^{(\alpha)\mu}\hat{F}^{(\beta)} - \lambda^{(\beta)\mu}\hat{F}^{(\alpha)}\right)\right)\right) \\ + \begin{pmatrix} -2\beta\hat{M}^{(\alpha\beta)\mu} + (2\beta+\alpha)\left(\hat{M}^{(\alpha)\mu(\beta)} - \hat{M}^{(\beta)\mu(\alpha)}\right) \\ +\alpha\left(\lambda^{(\alpha)\mu}\hat{M}^{(\beta)} - \lambda^{(\beta)\mu}\hat{M}^{(\alpha)}\right) \end{pmatrix} \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
\hat{R}^{(\alpha\beta)\mu\nu} &= \hat{\tilde{R}}^{(\alpha\beta)\mu\nu} + \hat{\hat{R}}^{(\alpha\beta)\mu\nu} \\
Q^{(\alpha\beta)\mu\nu} &= \tilde{Q}^{(\alpha\beta)\mu\nu} + \hat{Q}^{(\alpha\beta)\mu\nu} \\
D_\nu Q^{(\alpha\beta)\mu\nu} &= \tilde{D}_\nu Q^{(\alpha\beta)\mu\nu} + N_\nu Q^{(\alpha\beta)\mu\nu} + M^{(\alpha)}_{(\gamma)\nu} Q^{(\gamma\beta)\mu\nu} + M^{(\beta)}_{(\gamma)\nu} Q^{(\alpha\gamma)\mu\nu}
\end{aligned}$$

Therefore, according to the equation of motion of the spin field in the ECT spin-gravitational field theory:

$$D_\nu Q^{(\alpha\beta)\mu\nu} = \frac{8\pi\kappa}{c^4}\left(s_e^{(\alpha\beta)\mu} + s_g^{(\alpha\beta)\mu}\right)$$

$$\begin{Bmatrix} s_e^{(\alpha\beta)\mu} = \frac{c\hbar}{4}\bar{\psi}(\sigma^{(\alpha\beta)}\gamma^\mu + \gamma^\mu\sigma^{(\alpha\beta)})\psi \\ s_g^{(\alpha\beta)\mu} = -\frac{c^4}{16\pi G}\beta\left(K^{(\alpha)\mu(\beta)} - K^{(\beta)\mu(\alpha)}\right) + \frac{c^4}{16\pi G}\alpha\left(Q_E{}^{(\alpha)\mu(\beta)} - Q_E{}^{(\beta)\mu(\alpha)}\right) \end{Bmatrix}$$

We obtain the Riemannian background spacetime formulation of the equations of motion for the spin field as：

$$D_{\nu}Q^{(\alpha\beta)\mu\nu}=\frac{8\pi\kappa}{c^{4}}\left(s_{e}^{(\alpha\beta)\mu}+s_{g}^{(\alpha\beta)\mu}\right)$$

$$=\frac{8\pi\kappa}{c^{4}}s_{e}^{(\alpha\beta)\mu}+\frac{\kappa}{G}\begin{pmatrix}\left(\beta\widehat{K}^{\mu(\alpha\beta)}+\alpha\left(\widehat{F}^{\mu(\alpha\beta)}-\left(\lambda^{(\alpha)\mu}\widehat{F}^{(\beta)}-\lambda^{(\beta)\mu}\widehat{F}^{(\alpha)}\right)\right)\right)\\+\alpha\left(\widehat{M}^{(\alpha\beta)\mu}+\left(\lambda^{(\alpha)\mu}\widehat{M}^{(\beta)}-\lambda^{(\beta)\mu}\widehat{M}^{(\alpha)}\right)\right)\\-\left(2\beta+\alpha\right)\left(\widehat{M}^{(\alpha\beta)\mu}-\left(\widehat{M}^{(\alpha)\mu(\beta)}-\widehat{M}^{(\beta)\mu(\alpha)}\right)\right)\end{pmatrix}$$

$$\begin{pmatrix}\tilde{D}_{\nu}\widehat{Q}^{(\alpha\beta)\mu\nu}+N_{\nu}\widehat{Q}^{(\alpha\beta)\mu\nu}+\widehat{M}_{(\gamma)\nu}^{(\alpha)}\widehat{Q}^{(\gamma\beta)\mu\nu}+\widehat{M}_{(\gamma)\nu}^{(\beta)}\widehat{Q}^{(\alpha\gamma)\mu\nu}\\+\widehat{D}_{\nu}\tilde{Q}^{(\alpha\beta)\mu\nu}+N_{\nu}\tilde{Q}^{(\alpha\beta)\mu\nu}+M_{(\gamma)\nu}^{(\alpha)}\tilde{Q}^{(\gamma\beta)\mu\nu}+M_{(\gamma)\nu}^{(\beta)}\tilde{Q}^{(\alpha\gamma)\mu\nu}\\-\frac{\kappa}{G}\begin{pmatrix}-2\beta\widehat{M}^{(\alpha\beta)\mu}+\left(2\beta+\alpha\right)\left(\widehat{M}^{(\alpha)\mu(\beta)}-\widehat{M}^{(\beta)\mu(\alpha)}\right)\\+\alpha\left(\lambda^{(\alpha)\mu}\widehat{M}^{(\beta)}-\lambda^{(\beta)\mu}\widehat{M}^{(\alpha)}\right)\end{pmatrix}\end{pmatrix}=\begin{pmatrix}\frac{8\pi\kappa}{c^{4}}s_{e}^{(\alpha\beta)\mu}+\frac{\kappa}{G_{N}}\beta\widehat{K}^{\mu(\alpha\beta)}\\+\frac{\kappa}{G_{N}}\alpha\left(\widehat{F}^{\mu(\alpha\beta)}-\lambda^{(\alpha)\mu}\widehat{F}^{(\beta)}+\lambda^{(\beta)\mu}\widehat{F}^{(\alpha)}\right)\end{pmatrix}$$

......(3.22)

where：

$$s_{e}^{(\alpha\beta)\mu}=\frac{c\hbar}{4}\bar{\psi}(\sigma^{(\alpha\beta)}\gamma^{\mu}+\gamma^{\mu}\sigma^{(\alpha\beta)})\psi$$

Einstein-Tang Gravitational Field Equation of Motion

Because of:

$$G_{\sigma\nu}=R_{\sigma\nu}-\frac{1}{2}g_{\sigma\nu}R=\left(\tilde{R}_{\sigma\nu}-\frac{1}{2}\tilde{g}_{\sigma\nu}\tilde{R}\right)+\left(\widehat{R}_{\sigma\nu}-\frac{1}{2}g_{\sigma\nu}\widehat{R}\right)$$

$$2\hat{R}^{\mu(\alpha)}-\hat{R}\lambda^{(\alpha)\mu}=\left(2\hat{\tilde{R}}^{\mu(\alpha)}-\hat{\tilde{R}}\lambda^{(\alpha)\mu}\right)+\left(2\hat{\widehat{R}}^{\mu(\alpha)}-\hat{\widehat{R}}\lambda^{(\alpha)\mu}\right)$$

$$K_{\lambda\mu\nu}=\widehat{K}_{\lambda\mu\nu}-2\left(\widehat{M}_{\mu\nu\lambda}-\widehat{M}_{\mu\lambda\nu}+\widehat{M}_{\nu\lambda\mu}\right)$$

Therefore, according to the gravitational field equation of ECT spin-gravitational field theory [2]:

$$\left(R^{\mu\nu}-\frac{1}{2}g^{\mu\nu}R\right)+\frac{1}{2}\beta D_{\sigma}K^{\mu\nu\sigma}=\frac{8\pi G_{N}}{c^{4}}\begin{pmatrix}P_{e}^{\mu\nu}+P_{\gamma}^{\mu\nu}+P_{f}^{\mu\nu}+\bar{P}_{gk}^{\mu\nu}\\-\frac{c^{4}}{16\pi G_{N}}\alpha\left(2\hat{R}^{\nu\mu}-g^{\nu\mu}\hat{R}\right)\end{pmatrix}$$

We obtain the formulation of the gravitational field equation on the Riemannian background spacetime as:

$$\begin{pmatrix}\left(\tilde{R}^{\mu\nu}-\frac{1}{2}g^{\mu\nu}\tilde{R}\right)\\+\left(\widehat{R}^{\mu\nu}-\frac{1}{2}g^{\mu\nu}\widehat{R}\right)+\frac{1}{2}\beta\left(\tilde{D}_{\sigma}\widehat{K}^{\mu\nu\sigma}+N_{\sigma}\widehat{K}^{\mu\nu\sigma}\right)\\-\beta\begin{pmatrix}\tilde{D}_{\sigma}\left(\widehat{M}^{\mu\nu\sigma}-\widehat{M}^{\mu\sigma\nu}+\widehat{M}^{\nu\sigma\mu}\right)\\+N_{\sigma}\left(\widehat{M}^{\mu\nu\sigma}-\widehat{M}^{\mu\sigma\nu}+\widehat{M}^{\nu\sigma\mu}\right)\end{pmatrix}\end{pmatrix}=\frac{8\pi G_{N}}{c^{4}}\begin{pmatrix}P_{e}^{\mu\nu}+P_{\gamma}^{\mu\nu}+P_{f}^{\mu\nu}+\bar{P}_{gk}^{\mu\nu}\\-\frac{c^{4}}{16\pi G_{N}}\alpha\begin{pmatrix}\left(2\hat{\tilde{R}}^{\nu\mu}-g^{\nu\mu}\hat{\tilde{R}}\right)\\+\left(2\hat{\widehat{R}}^{\nu\mu}-g^{\nu\mu}\hat{\widehat{R}}\right)\end{pmatrix}\end{pmatrix}$$

where：

$$P_{e}^{(\alpha)\mu}=-\frac{1}{2}c\left(i\hbar\bar{\psi}\gamma^{\mu}D_{\rho}\psi-i\hbar D_{\rho}\bar{\psi}\gamma^{\mu}\psi\right)\lambda^{(\alpha)\rho}+L_{e}\lambda^{(\alpha)\mu}$$

$$P_{f}^{(\alpha)\mu}=\frac{c^{4}}{16\pi\kappa}\left(\hat{R}_{(\beta\gamma)\rho\sigma}Q^{(\beta\gamma)\mu\sigma}-\frac{1}{4}\delta_{\rho}^{\mu}\hat{R}_{(\beta\gamma)m\sigma}Q^{(\beta\gamma)m\sigma}\right)\lambda^{(\alpha)\rho}$$

$$P_{\gamma}^{(\alpha)\mu}=\frac{1}{4\pi}\left(F_{\rho\sigma}F^{\mu\sigma}-\frac{1}{4}\delta_{\rho}^{\mu}F_{m\sigma}F^{m\sigma}\right)\lambda^{(\alpha)\rho}$$

$$\overline{P}_{gk}^{(\alpha)\mu}=\frac{c^4}{32\pi G}\beta\left(-2F_{\rho m\sigma}K^{\mu m\sigma}+K_{(\beta)\sigma\rho}K^{(\beta)\sigma\mu}-\frac{1}{6}\delta_\rho^\mu K_{(\beta)m\sigma}K^{(\beta)m\sigma}\right)\lambda^{(\alpha)\rho}$$

(3.23)

The Equation of Motion for the Charge Current

$$\tilde{D}_\mu j_e^\mu+N_\mu j_e^\mu=0$$

……(3.24)

The Equation of Motion of the Spin Current

Because of

$$D_\mu s_e^{(\alpha\beta)\mu}=\tilde{D}_\mu s_e^{(\alpha\beta)\mu}+N_\mu s_e^{(\alpha\beta)\mu}+M_{(\gamma)\mu}^{(\alpha)}s_e^{(\gamma\beta)\mu}+M_{(\gamma)\mu}^{(\beta)}s_e^{(\alpha\gamma)\mu}$$

Therefore, according to the equation of motion of the spin current in the ECT spin- gravitational field theory

$$D_\mu s_e^{(\alpha\beta)\mu}=P_e^{(\beta\alpha)}-P_e^{(\alpha\beta)}$$

We obtain the equation of motion for the spin current on a Riemannian background spacetime as:

$$\tilde{D}_\mu s_e^{(\alpha\beta)\mu}+N_\mu s_e^{(\alpha\beta)\mu}+M_{(\gamma)\mu}^{(\alpha)}s_e^{(\gamma\beta)\mu}+M_{(\gamma)\mu}^{(\beta)}s_e^{(\alpha\gamma)\mu}=P_e^{(\beta\alpha)}-P_e^{(\alpha\beta)}$$

(3.25)

Energy-Momentum Current Equation of Motion

According to Energy-momentum current equation of motion in the ECT spin-gravitational field theory□

$$D_\nu P_e^{\mu\nu}=\left(-j_e^\nu F_{\nu\rho}+\frac{1}{2}s_e^{(\alpha\beta)\nu}\hat{R}_{(\alpha\beta)\nu\rho}\right)g^{\mu\rho}+\frac{1}{2}P_e^{\nu\sigma}H^\mu{}_{\nu\sigma}$$

We obtain the background-dependent formulation of the energy-momentum current equation of motion as:

$$\tilde{D}_\nu P_e^{\mu\nu}+N_{\sigma\nu}^\mu P_e^{\sigma\nu}+N_\nu P_e^{\mu\nu}=\begin{pmatrix}-j_e^\nu F_{\nu\rho}+\frac{1}{2}s_e^{(\alpha\beta)\nu}\hat{\hat{R}}_{(\alpha\beta)\nu\rho}\\+\frac{1}{2}s_e^{(\alpha\beta)\nu}\hat{\hat{R}}_{(\alpha\beta)\nu\rho}\end{pmatrix}g^{\mu\rho}+\frac{1}{2}P_e^{\nu\sigma}\left(\hat{H}_{\nu\sigma}^\mu+2M_{\nu\sigma}{}^\mu\right)$$

……(3.26)

Riemannian Spacetime Background Formulation of Lagrangian Density and Action

In the ECT spin-gravitional field theory [2], when $\beta_1=-\beta_2=1,\beta_3=0,\alpha''=0,\gamma=0$, there are:

（1）Lagrangian density of Dirac electron field:

$$\begin{aligned}L_e&=\frac{1}{2}c\left(i\hbar\overline{\psi}\gamma^\mu D_\mu\psi-i\hbar D_\mu\overline{\psi}\gamma^\mu\psi\right)-mc^2\overline{\psi}\psi\\&=\frac{1}{2}c\left(i\hbar\overline{\psi}\gamma^\mu\tilde{D}_\mu\psi-i\hbar\tilde{D}_\mu\overline{\psi}\gamma^\mu\psi\right)-mc^2\overline{\psi}\psi+\frac{1}{2}c\hbar\overline{\psi}\left(\gamma^\mu\Xi_\mu-\overline{\psi}\Xi_\mu\gamma^\mu\right)\psi\\&=\overline{L}_e+\frac{1}{2}s_e^{(\alpha\beta)\mu}\hat{\Gamma}_{(\alpha\beta)\mu}-j_e^\mu A_\mu\end{aligned}$$

……(3.27)

where:

$$\bar{L}_e = \frac{1}{2} c \left(i\hbar \bar{\psi}\gamma^{\mu} \tilde{D}_{\mu}\psi - i\hbar \tilde{D}_{\mu}\bar{\psi}\gamma^{\mu}\psi \right) - mc^2 \bar{\psi}\psi$$

$$s_e^{(\alpha\beta)\mu} = \frac{c\hbar}{4} \bar{\psi}(\sigma^{(\alpha\beta)}\gamma^{\mu} + \gamma^{\mu}\sigma^{(\alpha\beta)})\psi, \qquad ,$$

$$\left(s_e^{\rho\sigma\mu} = -s_e^{\sigma\rho\mu} = -s_e^{\rho\mu\sigma} \right)$$

$$j_e^{\mu} = e\bar{\psi}\gamma^{\mu}\psi, j^{\mu} = \bar{\psi}\gamma^{\mu}\psi$$

（2）Lagrangian density of Maxwell electromagnetic field：

$$L_{\gamma} = -\frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu}$$

......(3.28)

（3）Lagrangian density of Tangfeilin spin field：

$$\begin{aligned} L_f &= -\frac{c^4}{64\pi k} \hat{R}_{(\alpha\beta)\mu\nu} Q^{(\alpha\beta)\mu\nu} \\ &= -\frac{c^4}{64\pi k} \left(\hat{\tilde{R}}_{(\alpha\beta)\mu\nu} + \hat{\overset{\frown}{R}}_{(\alpha\beta)\mu\nu} \right) \left(\tilde{Q}^{(\alpha\beta)\mu\nu} + \overset{\frown}{Q}^{(\alpha\beta)\mu\nu} \right) \\ &= -\frac{c^4}{64\pi k} \left(\hat{\tilde{R}}_{(\alpha\beta)\mu\nu} \tilde{Q}^{(\alpha\beta)\mu\nu} + \hat{\tilde{R}}_{(\alpha\beta)\mu\nu} \overset{\frown}{Q}^{(\alpha\beta)\mu\nu} + \hat{\overset{\frown}{R}}_{(\alpha\beta)\mu\nu} \tilde{Q}^{(\alpha\beta)\mu\nu} + \hat{\overset{\frown}{R}}_{(\alpha\beta)\mu\nu} \overset{\frown}{Q}^{(\alpha\beta)\mu\nu} \right) \end{aligned}$$

...(3.29)

（3）Generalized Einstein gravitational field Lagrangian density (Einstein-Tang gravitational field Lagrangian density):：

$$\begin{aligned} L_g &= \frac{c^4}{16\pi G_N} \left(-2D_{\nu} F^{\nu} + \alpha \hat{R} + R - \frac{1}{4} \beta F_{\lambda\mu\nu} K^{\lambda\mu\nu} \right) \\ &= \frac{c^4}{16\pi G_N} \begin{pmatrix} -2D_{\nu} \left(\overset{\frown}{F}^{\nu} - \overset{\frown}{M}^{\nu} \right) + \alpha \hat{\tilde{R}} + \tilde{R} \\ + \alpha \hat{\overset{\frown}{R}} + \overset{\frown}{R} + \beta \left(-\frac{1}{4} \overset{\frown}{F}_{\lambda\mu\nu} \overset{\frown}{K}^{\lambda\mu\nu} + \overset{\frown}{M}_{\lambda\mu\nu} \overset{\frown}{F}^{\lambda\mu\nu} \right) \end{pmatrix} \end{aligned}$$

......(3.31)

The four-dimensional volume element is：

$$d\Omega = \sqrt{-g} d^4 x = |\omega| \sqrt{-\tilde{g}} d^4 x$$

$$|\omega| = \left| \omega_{\mu}{}^{\nu} \right|$$

...(3.31)

Therefore, the effective action is:

$$S = \frac{1}{c} \int L d\Omega = \sqrt{-g} d^4 x = \frac{1}{c} \int L |\omega| \sqrt{-\tilde{g}} d^4 x$$

(3.32)

Formulation in Flat Spacetime Background

Properties of Flat Spacetime

A Riemannian background spacetime whose tetrad basis vector $\vec{e}_{(\alpha)}, \vec{e}^{(\alpha)}$ is a constant vector is called a flat background spacetime. Thus, there are：

$$\tilde{D}_{\mu} \vec{e}_{(\alpha)} = \partial_{\mu} \vec{e}_{(\alpha)} = 0, \tilde{D}_{\mu} \vec{e}^{(\alpha)} = \partial_{\mu} \vec{e}^{(\alpha)} = 0$$

From the above equation, we obtain：

$$\hat{\tilde{\Gamma}}^{(\alpha)}_{(\beta)\mu} = 0$$

(4.1)

Therefore, we get:

$$\hat{\tilde{R}}^{(\alpha)}_{(\beta)\mu\nu} = 0, \tilde{R}^{\rho}_{\sigma\mu\nu} = 0$$

……(4.2)

Simplification of the Equations of Motion

Under the flat background space-time, according to the discussion in the previous section, we obtain the simplified ECT equation

- Simplified form of the equation of motion for the spin field:

$$\begin{pmatrix} \tilde{D}_\nu \hat{Q}^{(\alpha\beta)\mu\nu} + N_\nu \hat{Q}^{(\alpha\beta)\mu\nu} + \hat{M}^{(\alpha)}_{(\gamma)\nu}\hat{Q}^{(\gamma\beta)\mu\nu} + \hat{M}^{(\beta)}_{(\gamma)\nu}\hat{Q}^{(\alpha\gamma)\mu\nu} \\ -\frac{\kappa}{G}\begin{pmatrix} -2\beta\hat{M}^{(\alpha\beta)\mu} + (2\beta+\alpha)\left(\hat{M}^{(\alpha)\mu(\beta)} - \hat{M}^{(\beta)\mu(\alpha)}\right) \\ +\alpha\left(\lambda^{(\alpha)\mu}\hat{M}^{(\beta)} - \lambda^{(\beta)\mu}\hat{M}^{(\alpha)}\right) \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \frac{8\pi\kappa}{c^4} s_e^{(\alpha\beta)\mu} + \frac{\kappa}{G_N}\beta\hat{K}^{\mu(\alpha\beta)} \\ +\frac{\kappa}{G_N}\alpha\left(\hat{F}^{\mu(\alpha\beta)} - \lambda^{(\alpha)\mu}\hat{F}^{(\beta)} + \lambda^{(\beta)\mu}\hat{F}^{(\alpha)}\right) \end{pmatrix}$$

……(4.3)

- The simplified form of the equation of motion of the gravitational field:

$$\begin{pmatrix} \left(\hat{R}^{\mu\nu} - \frac{1}{2}g^{\mu\nu}\hat{R}\right) + \frac{1}{2}\beta\left(\tilde{D}_\sigma\hat{K}^{\mu\nu\sigma} + N_\sigma\hat{K}^{\mu\nu\sigma}\right) \\ -\beta\begin{pmatrix} \tilde{D}_\sigma\left(\hat{M}^{\mu\nu\sigma} - \hat{M}^{\mu\sigma\nu} + \hat{M}^{\nu\sigma\mu}\right) \\ +N_\sigma\left(\hat{M}^{\mu\nu\sigma} - \hat{M}^{\mu\sigma\nu} + \hat{M}^{\nu\sigma\mu}\right) \end{pmatrix} \end{pmatrix} = \frac{8\pi G_N}{c^4}\begin{pmatrix} P_e^{\mu\nu} + P_\gamma^{\mu\nu} + P_f^{\mu\nu} + \bar{P}_{gk}^{\mu\nu} \\ -\frac{c^4}{16\pi G_N}\alpha\left(2\hat{\hat{R}}^{\nu\mu} - g^{\nu\mu}\hat{\hat{R}}\right) \end{pmatrix}$$

……(4.4)

- Simplified form of the equation of motion for the charge current:

$$\tilde{D}_\mu j_e^\mu + N_\mu j_e^\mu = 0$$

……(4.5)

- Simplified form of the equation of motion of the spin current:

$$D_\mu s_e^{(\alpha\beta)\mu} + N_\mu s_e^{(\alpha\beta)\mu} + M^{(\alpha)}_{(\gamma)\mu}s_e^{(\gamma\beta)\mu} + M^{(\beta)}_{(\gamma)\mu}s_e^{(\alpha\gamma)\mu} = P_e^{(\beta\alpha)} - P_e^{(\alpha\beta)}$$

……(4.6)

- Simplified form of the equation of motion of the energy-momentum current:

$$\tilde{D}_\nu P_e^{\mu\nu} + N^\mu_{\sigma\nu}P_e^{\sigma\nu} + N_\nu P_e^{\mu\nu} = \left(-j_e^\nu F_{\nu\rho} + \frac{1}{2}s_e^{(\alpha\beta)\nu}\hat{\hat{R}}_{(\alpha\beta)\nu\rho}\right)g^{\mu\rho} + \frac{1}{2}P_e^{\nu\sigma}\left(\hat{H}^\mu_{\nu\sigma} + 2\hat{M}_{\nu\sigma}{}^\mu\right)$$

…(4.7)

Simplification of Lagrangian Density

Simplified form of the Lagrangian density:

$$L_e = \bar{L}_e + \frac{1}{2}s_e^{(\alpha\beta)\mu}\hat{\Gamma}_{(\alpha\beta)\mu} - j_e^\mu A_\mu$$

$$\text{where}\quad\begin{pmatrix} \bar{L}_e = \frac{1}{2}c\left(i\hbar\bar{\psi}\gamma^\mu\tilde{D}_\mu\psi - i\hbar\tilde{D}_\mu\bar{\psi}\gamma^\mu\psi\right) - mc^2\bar{\psi}\psi \\ s_e^{(\alpha\beta)\mu} = \frac{c\hbar}{4}\bar{\psi}(\sigma^{(\alpha\beta)}\gamma^\mu + \gamma^\mu\sigma^{(\alpha\beta)})\psi, \left(s_e^{\rho\sigma\mu} = -s_e^{\sigma\rho\mu} = -s_e^{\rho\mu\sigma}\right) \\ j_e^\mu = e\bar{\psi}\gamma^\mu\psi, j^\mu = \bar{\psi}\gamma^\mu\psi \end{pmatrix}$$

$$L_\gamma = -\frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu}$$
$$L_f = -\frac{c^4}{64\pi k} \hat{\hat{R}}_{(\alpha\beta)\mu\nu} \hat{Q}^{(\alpha\beta)\mu\nu}$$
$$L_g = \frac{c^4}{16\pi G_N}\left(-2D_\nu\left(\hat{F}^\nu - \hat{M}^\nu\right) + \alpha\hat{\hat{R}} + \hat{R} + \beta\left(-\frac{1}{4}\hat{F}_{\lambda\mu\nu}\hat{K}^{\lambda\mu\nu} + \hat{M}_{\lambda\mu\nu}\hat{F}^{\lambda\mu\nu}\right)\right)$$

……(4.8)

1 + 3 Decomposition in Inertial Reference System

Flat background space-time can adopt time-axis separated coordinate system (x^0, x^i). The time-axis separated coordinate system of flat spacetime is actually an inertial reference system. In the time-axis separated coordinate system, the flat background spacetime metric tensor can be written in the form：

$$\eta_{(\alpha\beta)} = \begin{pmatrix} 1 & 0 \\ 0 & -\delta_{(ab)} \end{pmatrix}, \eta^{(\alpha\beta)} = \begin{pmatrix} 1 & 0 \\ 0 & -\delta^{(ab)} \end{pmatrix}, \tilde{g}_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & -\tilde{g}_{ij} \end{pmatrix}, \tilde{g}^{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & -\tilde{g}^{ij} \end{pmatrix}$$
$$\tilde{\lambda}^{(\alpha)}_\mu = \begin{pmatrix} 1 & 0 \\ 0 & \tilde{\lambda}^{(a)}_j \end{pmatrix}, \tilde{\lambda}_{(\alpha)\mu} = \begin{pmatrix} 1 & 0 \\ 0 & -\tilde{\lambda}_{(a)j} \end{pmatrix}, \tilde{\lambda}^\mu_{(\alpha)} = \begin{pmatrix} 1 & 0 \\ 0 & \tilde{\lambda}^j_{(a)} \end{pmatrix}, \tilde{\lambda}^{(\alpha)\mu} = \begin{pmatrix} 1 & 0 \\ 0 & -\tilde{\lambda}^{(a)j} \end{pmatrix}$$
$$\delta_{(ab)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \delta^{(ab)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

……(4.9)

Where $,\tilde{\lambda}^{(\alpha)}_\mu, \tilde{\lambda}^\mu_{(\alpha)}, \tilde{g}_{\mu\nu}, \tilde{g}^{\mu\nu}$ is only related to the spatial coordinate x^i and is independent of the time coordinate $t, (x^0 = ct)$, $a, b, i, j = 1, 2, 3$ is that correspond spatial indicator.

The only affine connections $\tilde{\Gamma}^\rho_{\sigma\mu}$ that are not zero in this case are：

$$\tilde{\Gamma}^k_{ij} = \frac{1}{2}\tilde{g}^{kl}\left(\partial_i \tilde{g}_{lj} + \partial_j \tilde{g}_{li} - \partial_l \tilde{g}_{ij}\right)$$

……(4.10)

Time is a good spatial scalar in the time-axis separated coordinate system of flat background spacetime.

In curved spacetime with torsion, the ADM decomposition of the metric are [3,4]:

$$g_{\mu\nu} = \begin{pmatrix} N^2 - N_k N^k & -N_j \\ -N_i & -g_{ij} \end{pmatrix}, g^{\mu\nu} = \begin{pmatrix} \frac{1}{N^2} & -\frac{N^j}{N} \\ -\frac{N^i}{N} & -g^{ij} \end{pmatrix}$$
$$g_{ij} = g_{ji}, g^{ik} g_{kj} = \delta^i_j, N_i = g_{ij} N^j, N^i = g^{ij} N_j$$

The g_{ij}, g^{ij} in the ADM decomposition above is the 3-dimensional spatial metric. Let $\lambda^{(a)}_i, \lambda^i_{(a)}$ be the tetrad field of 3-dimensional space, $\delta_{(ab)}, \delta^{(ab)}$ is the tetrad metric of 3-dimensional space, and $\omega_i{}^j, \varpi_i{}^j$ is the curved deviation tensor of 3-dimensional space, there are:

$$\lambda^{(a)}_i = \omega_i{}^j \tilde{\lambda}^{(a)}_j, \lambda^i_{(a)} = \tilde{\lambda}^j_{(a)} \varpi_j{}^i$$
$$\omega_i{}^k \varpi_k{}^j = \varpi_i{}^k \omega_k{}^j = \delta^j_i$$
$$g_{ij} = \tilde{g}_{mn} \omega_i{}^m \omega_j{}^n, g^{ij} = \tilde{g}^{mn} \varpi_m{}^j \varpi_n{}^j,$$

$$\lambda^{i}_{(a)} = g^{ij}\delta_{(ab)}\lambda^{(b)}_{j}, \lambda^{(a)}_{i} = g_{ij}\delta^{(ab)}\lambda^{j}_{(b)}$$

......(4.11)

Therefore, the ADM decomposition of the tetrad field in curved spacetime with torsion is:

$$\lambda^{(\alpha)}_{\mu} = \begin{pmatrix} N & N^{k}\lambda^{(a)}_{k} \\ 0 & \lambda^{(a)}_{i} \end{pmatrix}$$

Solved by $\lambda^{(\alpha)}_{\mu} = \omega_{\mu}{}^{\nu}\tilde{\lambda}^{(\alpha)}_{\nu}, \lambda^{\mu}_{(\alpha)} = \lambda^{(\alpha)}_{\nu}\varpi_{\nu}{}^{\mu}, \varpi_{\mu}{}^{\rho}\omega_{\rho}{}^{\nu} = \varpi_{\rho}{}^{\nu}\omega_{\mu}{}^{\rho} = \delta^{\nu}_{\mu}$:

$$\omega_{\mu}{}^{\nu} = \begin{pmatrix} N & N^{k}\omega_{k}{}^{j} \\ 0 & \omega_{i}{}^{j} \end{pmatrix}, \varpi_{\mu}{}^{\nu} = \begin{pmatrix} \frac{1}{N} & -\frac{N^{j}}{N} \\ 0 & \varpi_{i}{}^{j} \end{pmatrix}$$

This is the ADM decomposition of the tetrad field of a curved spacetime with torsion on a flat background spacetime.

For the generalized ADM decomposition (TFL-ADM decomposition) in the most general case, we can set:

$$\omega_{\mu}{}^{\nu} = \begin{pmatrix} N & N^{k}\omega_{k}{}^{j} \\ \bar{N}_{i} & \omega_{i}{}^{j} \end{pmatrix}$$

......(4.12)

In this case, it can be solved by $\varpi_{\mu}{}^{\rho}\omega_{\rho}{}^{\nu} = \varpi_{\rho}{}^{\nu}\omega_{\mu}{}^{\rho} = \delta^{\nu}_{\mu}$:

$$\varpi_{\mu}{}^{\nu} = \begin{pmatrix} \frac{1}{N - N^{m}\bar{N}_{m}} & -\frac{N^{j}}{N - N^{m}\bar{N}_{m}} \\ -\frac{\varpi_{i}{}^{k}\bar{N}_{k}}{N - N^{m}\bar{N}_{m}} & \varpi_{i}{}^{j} + \frac{\varpi_{i}{}^{k}\bar{N}_{k}N^{j}}{N - N^{m}\bar{N}_{m}} \end{pmatrix}$$

......(4.13)

This is the generalized ADM decomposition (or TFL-ADM decomposition) of the tetrad field of a curved spacetime with torsion on a flat background spacetime. The metric tensor is:

$$\begin{aligned} g_{\mu\nu} &= \omega_{\mu}{}^{\rho}\tilde{g}_{\rho\sigma}\omega_{\nu}{}^{\sigma} = wGw^{T} \\ &= \begin{pmatrix} N & N^{k}\omega_{k}{}^{m} \\ \bar{N}_{i} & \omega_{i}{}^{m} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -\tilde{g}_{mn} \end{pmatrix}\begin{pmatrix} N & \bar{N}_{j} \\ N^{k}\omega_{k}{}^{n} & \omega_{j}{}^{n} \end{pmatrix} \\ &= \begin{pmatrix} N & -N^{k}\omega_{k}{}^{m}\tilde{g}_{mn} \\ \bar{N}_{i} & -\omega_{i}{}^{m}\tilde{g}_{mn} \end{pmatrix}\begin{pmatrix} N & \bar{N}_{j} \\ N^{k}\omega_{k}{}^{n} & \omega_{j}{}^{n} \end{pmatrix} \\ &= \begin{pmatrix} N^{2} - N^{k}\omega_{k}{}^{m}\tilde{g}_{mn}N^{k'}\omega_{k'}{}^{n} & N\bar{N}_{j} - N^{k}\omega_{k}{}^{m}\tilde{g}_{mn}\omega_{j}{}^{n} \\ N\bar{N}_{i} - \omega_{i}{}^{m}\tilde{g}_{mn}N^{k}\omega_{k}{}^{n} & \bar{N}_{i}\bar{N}_{j} - \omega_{i}{}^{m}\tilde{g}_{mn}\omega_{j}{}^{n} \end{pmatrix} \\ &= \begin{pmatrix} N^{2} - N^{m}g_{mn}N^{n} & N\bar{N}_{j} - N^{k}g_{kj} \\ N\bar{N}_{i} - g_{ik}N^{k} & \bar{N}_{i}\bar{N}_{j} - g_{ij} \end{pmatrix} \end{aligned}$$

......(4.14)

Formulation in the Background of Generalized ADM Spacetimes

Generalized ADM Decomposition of Tetrad Field

According to the decomposition formula of the tetrad field in flat spacetime in the previous section, we get:

$$\lambda_\mu^{(\alpha)} = \omega_\mu{}^\nu \tilde{\lambda}_\nu^{(\alpha)} = \tilde{\lambda}\omega^T$$
$$= \begin{pmatrix} 1 & 0 \\ 0 & \tilde{\lambda}_m^{(a)} \end{pmatrix} \begin{pmatrix} N & \bar{N}_i \\ N^k \omega_k{}^m & \omega_i{}^m \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 \\ 0 & \tilde{\lambda}_m^{(a)} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \omega_k{}^m \end{pmatrix} \begin{pmatrix} N & \bar{N}_i \\ N^k & \delta_i^k \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 \\ 0 & \tilde{\lambda}_m^{(a)} \omega_k{}^m \end{pmatrix} \begin{pmatrix} N & \bar{N}_i \\ N^k & \delta_i^k \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 \\ 0 & \lambda_k^{(a)} \end{pmatrix} \begin{pmatrix} N & \bar{N}_i \\ N^k & \delta_i^k \end{pmatrix}$$

......(5.1)

therefore, the generalized ADM decomposition of the tetrad field $\lambda_\mu^{(\alpha)}$ in curved spacetime with torsion (TFL-ADM decomposition) is:

$$\tilde{\lambda}_\mu^{(\alpha)} = \begin{pmatrix} 1 & 0 \\ 0 & \lambda_i^{(a)} \end{pmatrix}, \omega_\mu{}^\nu = \begin{pmatrix} N & N^j \\ \bar{N}_i & \delta_i^j \end{pmatrix}$$

......(5.2)

Solved by $\varpi_\mu{}^\rho \omega_\rho{}^\nu = \varpi_\rho{}^\nu \omega_\mu{}^\rho = \delta_\mu^\nu$:

$$\varpi_\mu{}^\nu = \begin{pmatrix} \dfrac{1}{N - N^m \bar{N}_m} & -\dfrac{N^j}{N - N^m \bar{N}_m} \\ -\dfrac{\bar{N}_i}{N - N^m \bar{N}_m} & \delta_i^j + \dfrac{\bar{N}_i N^j}{N - N^m \bar{N}_m} \end{pmatrix}$$

......(5.3)

Solved by $\tilde{\lambda}_\mu^{(\alpha)} \tilde{\lambda}_{(\alpha)}^\nu = \delta_\mu^\nu, \tilde{\lambda}_\rho^{(\alpha)} \tilde{\lambda}_{(\beta)}^\rho = \delta_{(\beta)}^{(\alpha)}$:

$$\tilde{\lambda}_{(\alpha)}^\mu = \begin{pmatrix} 1 & 0 \\ 0 & \lambda_{(a)}^i \end{pmatrix}$$

......(5.4)

Where $\lambda_i^{(a)}, \lambda_{(a)}^i; g_{ij}, g^{ij}; \delta_{(ab)}, \delta^{(ab)}$ is the tetrad field, coordinate metric, and tetrad metric of the 3-dimensional space, respectively. And there are:

$$g_{ij} = \lambda_i^{(a)} \delta_{(ab)} \lambda_j^{(b)}, g^{ij} = \lambda_{(a)}^i \delta^{(ab)} \lambda_{(b)}^j$$
$$\lambda_{(a)}^i = g^{ij} \delta_{(ab)} \lambda_j^{(b)}, \lambda_i^{(a)} = g_{ij} \delta^{(ab)} \lambda_{(b)}^j$$
$$g^{ik} g_{kj} = \delta_j^i, g_{ij} = g_{ji}, g^{ij} = g^{ji}$$

......(5.5)

Generalized ADM Space-Time Properties

According to the tetrad field formula $\tilde{\lambda}_\mu^{(\alpha)} = \begin{pmatrix} 1 & 0 \\ 0 & \lambda_i^{(a)} \end{pmatrix}, \tilde{D}_\nu \tilde{\lambda}_\mu^{(\alpha)} = 0$ of ADM space-time background, we obtain the non-zero four-dimensional affine connection formula and curvature tensor formula as follows:

$$\tilde{\Gamma}_{ij}^t = -\frac{1}{2} \partial_t g_{ij}, \tilde{\Gamma}_{tj}^i = -\frac{1}{2} g^{ik} \partial_t g_{kj}, \tilde{\Gamma}_{ij}^i = \frac{1}{2} g^{ik} \left(\partial_i g_{kj} + \partial_k g_{ij} - \partial_k g_{ij} \right)$$
$$\hat{\tilde{\Gamma}}_{(a)i}^{(0)} = \partial_t \lambda_{(a)i}, \hat{\tilde{\Gamma}}_{(b)t}^{(a)} = \frac{1}{2} \left(\lambda^{(a)i} \partial_t \lambda_{(b)i} - \lambda_{(b)}^i \partial_t \lambda_i^{(a)} \right), \hat{\tilde{\Gamma}}_{(b)i}^{(a)} = \frac{1}{2} \tilde{H}_{i(cb)} \delta^{(ca)}$$
$$\text{where} \quad \tilde{H}_{\mu(ab)} = \tilde{F}_{\mu(ab)} + \tilde{F}_{(a)\mu(b)} - \tilde{F}_{(b)\mu(a)}, \tilde{F}_{ij}^{(a)} = \partial_i \lambda_j^{(a)} - \partial_j \lambda_i^{(a)}$$

$$\hat{\tilde{R}}^{(0)}_{(a)ti} = \partial_t \hat{\tilde{\Gamma}}^{(0)}_{(a)i} - \hat{\tilde{\Gamma}}^{(0)}_{(c)i} \hat{\tilde{\Gamma}}^{(c)}_{(a)t}, \hat{\tilde{R}}^{(0)}_{(a)ij} = \partial_i \hat{\tilde{\Gamma}}^{(0)}_{(a)j} - \partial_j \hat{\tilde{\Gamma}}^{(0)}_{(a)i} + \hat{\tilde{\Gamma}}^{(0)}_{(c)i} \hat{\tilde{\Gamma}}^{(c)}_{(a)j} - \hat{\tilde{\Gamma}}^{(0)}_{(c)j} \hat{\tilde{\Gamma}}^{(c)}_{(a)i}$$
$$\hat{\tilde{R}}^{(a)}_{(b)ti} = \partial_t \hat{\tilde{\Gamma}}^{(a)}_{(b)i} + \hat{\tilde{\Gamma}}^{(a)}_{(c)t} \hat{\tilde{\Gamma}}^{(c)}_{(b)i} - \hat{\tilde{\Gamma}}^{(a)}_{(c)i} \hat{\tilde{\Gamma}}^{(c)}_{(b)t}$$
$$\hat{\tilde{R}}^{(a)}_{(b)ij} = \partial_i \hat{\tilde{\Gamma}}^{(a)}_{(b)j} - \partial_j \hat{\tilde{\Gamma}}^{(a)}_{(b)i} + \hat{\tilde{\Gamma}}^{(a)}_{(c)i} \hat{\tilde{\Gamma}}^{(c)}_{(b)j} - \hat{\tilde{\Gamma}}^{(a)}_{(c)j} \hat{\tilde{\Gamma}}^{(c)}_{(b)i} + \hat{\tilde{\Gamma}}^{(a)}_{(0)i} \hat{\tilde{\Gamma}}^{(0)}_{(b)j} - \hat{\tilde{\Gamma}}^{(a)}_{(0)j} \hat{\tilde{\Gamma}}^{(0)}_{(b)i}$$
$$\tilde{R}^{\rho}_{\sigma\mu\nu} = \hat{\tilde{R}}^{\rho}_{\sigma\mu\nu} = \tilde{\lambda}^{\rho}_{(\alpha)} \tilde{\lambda}^{(\beta)}_{\sigma} \hat{\tilde{R}}^{(\alpha)}_{(\beta)\mu\nu}$$

……(5.6)

Dynamic Field Properties

The non-zero dynamical field $\widehat{F}^{(\alpha)}_{\mu\nu}$ are: $\widehat{F}^{(a)}_{\mu\nu}$

Formulation in the Background of Einstein Spacetime

According to Einstein, the gravitational field is a geometric effect of Riemannian spacetime, and other force fields are dynamical effects, so the decomposition of the spin-gravitational field in the background of Einstein spacetime should be:

$$\omega_{\mu}^{\ \nu} = \varpi_{\mu}^{\ \nu} = \delta^{\nu}_{\mu}$$

……(6.1)

At this time, there are:

$$\widehat{F}_{\lambda\mu\nu} = 0$$

……(6.2)

Einstein's theory of gravity becomes $R + R^2$ higher order theory of gravity.

Covariant Gauge Condition

Because the action of the spin-gravitational field is invariant under the transformation of any coordinate system $x'^{\mu} = f^{\mu}\left(x^{\nu}\right)$ and the transformation of the Poincare group ($GL(1,3) \otimes SO(1,3)$) with respect to the tetrad spacetime, the spin-gravitational field has 4 + 4 + 6 gauge conditions. The gauge conditions are divided into 4-coordinate gauge conditions, four 4-tetrad translation gauge conditions and 6-Lorentz group gauge conditions. The spin field corresponds to 6-Lorentz group gauge conditions, and the gravitational field corresponds to 4-coordinate gauge conditions and 4-tetrad translation gauge conditions. The electromagnetic field has the invariance under the U(1) group transformation, so the electromagnetic field has 1-gauge condition.

Covariant Gauge Condition of Gravitational Field

Since the gravitational field has 8-gauge conditions, in the Formulation of the Riemannian space-time background, the gravitational field describes the physical effects of gravity by the curved deviation tensor $\omega_{\mu}^{\ \nu}$, so we can let the gauge conditions of the gravitational field be:

$$\tilde{D}^{\rho}\omega_{\rho\mu} = \tilde{D}^{\rho}\omega_{\mu\rho} = k\tilde{D}_{\mu}\omega, \omega = \omega_{\rho}^{\ \rho}$$

(7.1)

Where k is an undetermined constant and is obtained from the simplest form of the equations of motion.

We can separate the curved deviation tensor $\omega_{\mu}^{\ \nu}$ into symmetric $\phi_{\mu\nu}$ and antisymmetric $\chi_{\mu\nu}$ parts in the following way:

$$\omega_{\mu}^{\ \nu} = \delta^{\nu}_{\mu} + h_{\mu}^{\ \nu} = \delta^{\nu}_{\mu} + \phi_{\mu}^{\ \nu} + \chi_{\mu}^{\ \nu}$$
$$\phi_{\mu\nu} = \phi_{\nu\mu}, \chi_{\mu\nu} = -\chi_{\nu\mu}, h_{\mu\nu} = \phi_{\mu\nu} + \chi_{\mu\nu}$$

So, we get:

$$\phi_{\mu\nu} = \frac{1}{2}\left(\omega_{\mu\nu} + \omega_{\nu\mu}\right) - \tilde{g}_{\mu\nu}, \chi_{\mu\nu} = \frac{1}{2}\left(\omega_{\mu\nu} - \omega_{\nu\mu}\right)$$

The gravitational field gauge condition then become:

$$\tilde{D}^{\rho}\phi_{\rho\mu} = k\tilde{D}_{\mu}\phi, \tilde{D}^{\rho}\chi_{\rho\mu} = 0, \phi = \phi_{\rho}{}^{\rho}$$

(7.2)

The gravitational field symmetric deviation tensor $\phi_{\mu\nu}$ has 6-independent components, and the gravitational field antisymmetric deviation tensor $\chi_{\mu\nu}$ has 2-independent components.

The gravitational field gauge condition can also be expressed by $h_{\mu\nu}$ as:

$$\tilde{D}^{\rho}h_{\rho\mu} = \tilde{D}^{\rho}h_{\mu\rho} = k\tilde{D}^{\mu}h, h = h_{\rho}{}^{\rho}$$

...(7.3)

Covariant Gauge Condition of Spin Field

Since the spin field has 6-gauge conditions, in the background picture Formulation, the physical effect of the spin field is characterized by the twisted deviation tensor $M^{(\alpha)}_{(\beta)\mu}$, so we can let the gauge condition of the spin field be:

$$\tilde{D}^{\rho}\left(M_{(\alpha\beta)\rho} + a\left(M_{(\alpha)\rho(\beta)} - M_{(\beta)\rho(\alpha)}\right) + b\left(\tilde{\lambda}_{(\alpha)\mu}M_{(\beta)} - \tilde{\lambda}_{(\beta)\mu}M_{(\alpha)}\right)\right) = 0$$

......(7.4)

Where a, b is chosen such that the equations of motion for the spin field are the simplest. It is similar to the Lorentz gauge condition of the electromagnetic field, so it can be called the Lorentz gauge condition of the spin field.

Covariant Gauge Condition of Electromagnetic Field

Since the electromagnetic field has 1-gauge condition, we can let the gauge condition of the electromagnetic field be:

$$\tilde{D}^{\rho}A_{\rho} = 0$$

......(7.5)

It is known as the Lorentz gauge condition, which is called the Lorentz gauge condition of the electromagnetic field.

Non-Covariant Gauge Condition

Gauge Condition for Spatial Isotropy of Gravitational Field

Time is a good spatial scalar in the time-axis separated coordinate system of flat background space-time. Therefore, the gauge condition of spatial homogeneity can be established in the flat background space-time.

There are 8-gauge conditions for the curved deviation tensor $\omega_{\nu}{}^{\mu}, \varpi_{\mu}{}^{\nu}$ describing the gravitational field, so we can assume:

$$N = \Phi, \omega_i{}^j = \Omega\delta_i^j, \bar{N}_i = \bar{h}_i, N^i = \frac{h^i}{\Omega}$$

As a gauge condition for the gravitational field, there is:

$$\omega_{\mu}{}^{\nu} = \begin{pmatrix} \Phi & h^j \\ \bar{h}_i & \Omega\delta_i^j \end{pmatrix}, \varpi_{\mu}{}^{\nu} = \begin{pmatrix} \dfrac{\Omega}{\Phi\Omega - \bar{h}_k h^k} & -\dfrac{h^j}{\Phi\Omega - \bar{h}_k h^k} \\ -\dfrac{\bar{h}_i}{\Phi\Omega - \bar{h}_k h^k} & \dfrac{1}{\Omega}\left(\delta_i^j + \dfrac{\bar{h}_i h^j}{\Phi\Omega - \bar{h}_k h^k}\right) \end{pmatrix}$$

......(7.6)

Where $i, j = 1,2,3$ is the corresponding spatial index, Φ, Ω is a three-dimensional spatial scalar, and $h^i, \bar{h}_j$ is a three-dimensional spatial vector. Because the pure spatial part of $\omega_{\mu}{}^{\nu}$ is isotropic, we call this gauge condition the isotropic gauge condition of the gravitational field. The above isotropic gauge condition of the gravitational field can also be used as the gauge conditions for the gravitational field in an arbitrary background spacetime. Therefore, the spacetime metric of a curved spacetime with torsion is:

$$
\begin{aligned}
g_{\mu\nu} &= \omega_{\mu}{}^{\rho}\tilde{g}_{\rho\sigma}\omega_{\nu}{}^{\sigma} = wGw^{T} \\
&= \begin{pmatrix} \Phi & h^{m} \\ \bar{h}_{i} & \Omega\delta_{i}^{m} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -\tilde{g}_{mn} \end{pmatrix} \begin{pmatrix} \Phi & \bar{h}_{j} \\ h^{n} & \Omega\delta_{j}^{n} \end{pmatrix} \\
&= \begin{pmatrix} \Phi^{2} - \tilde{g}_{mn}h^{m}h^{n} & \Phi\bar{h}_{j} - \Omega h_{j} \\ \Phi\bar{h}_{i} - \Omega h_{i} & -\Omega^{2}\tilde{g}_{ij} + \bar{h}_{i}\bar{h}_{j} \end{pmatrix}
\end{aligned}
$$

(7.7)

The determinant of the deviation tensor is:

$$
\begin{aligned}
|\omega| &= \left|\omega_{\mu}{}^{\nu}\right| = \Omega^{2}\left(\Phi\Omega - \bar{h}_{k}h^{k}\right) \\
|\varpi| &= \left|\varpi_{\mu}{}^{\nu}\right| = \frac{1}{\Omega^{2}\left(\Phi\Omega - \bar{h}_{k}h^{k}\right)}
\end{aligned}
$$

(7.8)

The contraction of the deviation tensor $\omega_{\mu}{}^{\nu}, \varpi_{\mu}{}^{\nu}$ is:

$$
\begin{aligned}
\omega &= \omega_{\mu}{}^{\mu} = \Phi + 3\Omega \\
\varpi &= \varpi_{\mu}{}^{\mu} = \frac{1}{\Omega}\left(3 + \frac{\Omega^{2} + \bar{h}_{k}h^{k}}{\Phi\Omega - \bar{h}_{k}h^{k}}\right)
\end{aligned}
$$

(7.9)

The Gauge Conditions in Time-Separated Coordinate System

In the generalized ADM decomposition, the metric tensor of the Riemannian background spacetime is:

$$
g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & -g_{ij} \end{pmatrix}
$$

In this case, the time and space of the Riemannian background spacetime are separated, and a unified time can be used, in which the time-axis separated coordinate system is actually the inertial coordinate system under the curved space background, and we can set gauge conditions for the time component or space component of the field variables in this time-axis separated coordinate system.

Time-Dependent Gauge Condition of Spin Field

There are 6-gauge conditions for the spin field, similar to the SU (n) gauge field [5-7], and the gauge conditions can be set in the time-axis separated coordinate system as:

$$
M_{(\alpha\beta)0} + a\left(M_{(\alpha)0(\beta)} - M_{(\beta)0(\alpha)}\right) + b\left(\tilde{\lambda}_{(\alpha)0}M_{(\beta)} - \tilde{\lambda}_{(\beta)0}M_{(\alpha)}\right) = 0
$$

……(8.1)

Radiation Gauge Condition of Spin Field

When the gravitational field adopts the time axis separation gauge condition, the 6-gauge conditions of the spin field can be expressed as:

$$
\tilde{D}^{i}\left(M_{(\alpha\beta)i} + a\left(M_{(\alpha)i(\beta)} - M_{(\beta)i(\alpha)}\right) + b\left(\tilde{\lambda}_{(\alpha)i}M_{(\beta)} - \tilde{\lambda}_{(\beta)i}M_{(\alpha)}\right)\right) = 0
$$

……(8.2)

Time-Dependent Gauge Condition of Electromagnetic Field

The electromagnetic field has 1-gauge condition, so the gauge condition can be set in the time-axis separated coordinate system as follows[5-7]:

$$A_0 = 0$$

……(8.3)

Radiation Gauge Condition of Electromagnetic Field

When the gravitational field adopts the time axis separation gauge condition, the 1-gauge conditions of the electromagnetic field can be expressed as[5-7]:

$$\tilde{D}^i A_i = 0$$

……(8.4)

Linear Approximation of the Field Equation of Motion

According to the above discussion, the curved deviation tensor $\omega_{\mu\nu}$ can be decomposed into a symmetric curved deviation tensor $\phi_{\mu\nu}$ and an anti-symmetric curved deviation tensor $\chi_{\mu\nu}$. The number of gauge conditions of the symmetric curved deviation tensor $\phi_{\mu\nu}$ is the same as that of the potential-metric components in Einstein gravity theory. Therefore, the number of gauge conditions for the symmetric curved deviation tensor is consistent with the number of gauge conditions for the gravitational potential-metric in Einstein theory of gravity. Therefore, the symmetric deviation tensor $\phi_{\mu\nu}$ in the ECT spin-gravity field theory is equivalent to the gravitational potential in the Einstein theory of gravity. The ECT spin-gravitational field theory is an extension of Einstein gravitational theory by adding the antisymmetric curevd deviation tensor $\chi_{\mu\nu}$ and the twisted deviation tensor $M^{\rho}_{\sigma\mu}$. These deviation tensors are related to the excitation source by:

$\phi_{\mu\nu}$ --- Symmetrical part of energy-momentum current, $\chi_{\mu\nu}$ --- The antisymmetric part of that energy-momentum current, $M^{\rho}_{\sigma\mu}$ --- Spin current

Due to the existence of the spin equation of motion (conservation of angular momentum), the antisymmetric part of the energy-momentum current is equivalent to the rate of change of the spin current in space-time.In fact, the excitation source is the space-time rate of change of the spin current. Therefore, in fact, the excitation source of $\chi_{\mu\nu}$ is the space-time rate of change of the spin current.

At the same time, $\phi_{\mu\nu}, \chi_{\mu\nu}, M^{\rho}_{\sigma\mu}$ excites each other.Together, they make up the generalized gravitational field: the spin-gravity field. Therefore, the linear approximation of the ECT spin-gravitational field theory is to obtain the linear approximation equation for $\phi_{\mu\nu}, \chi_{\mu\nu}, M_{\rho\sigma\mu}$. The following only considers the simplified ECT theory (SECT theory) under the $\begin{pmatrix} \beta_1 = -\beta_2 = 1,\ \beta_3 = 0,\ \alpha'' = 0,\ \gamma = 0 \\ k_1 = 1, k_2 = -k_3 = 0, k_4 = -k_5 = k_6 = -\dfrac{1}{2} \end{pmatrix}$ condition.

Linear Approximation of Field Physical Quantities

We consider the case of a weak field in a flat background spacetime, in which the physical quantities of the field are expanded in series with a given small parameter ϵ:

$$\omega_\mu{}^\nu = \delta_\mu^\nu + h_\mu{}^\nu, \varpi_\mu{}^\nu = \delta_\mu^\nu + \bar{h}_\mu{}^\nu$$
$$\bar{h}_{\mu\nu} = -h_{\mu\nu} + O(\varepsilon), h_{\mu\nu} = \phi_{\mu\nu} + \chi_{\mu\nu}$$
$$\phi_{\mu\nu} = \frac{1}{2}\left(h_{\mu\nu} + h_{\nu\mu}\right), \chi_{\mu\nu} = \frac{1}{2}\left(h_{\mu\nu} - h_{\nu\mu}\right)$$
$$\psi \approx \varepsilon^{\frac{1}{2}}\left(\psi + O(\varepsilon)\right), \bar{\psi} \approx \varepsilon^{\frac{1}{2}}\left(\bar{\psi} + O(\varepsilon)\right)$$
$$A_\mu \approx \varepsilon A_\mu + O(\varepsilon^2), \hat{M}^\rho_{\sigma\mu} \approx \varepsilon M^\rho_{\sigma\mu} + O(\varepsilon^2)$$
$$\phi_{\mu\nu} \approx \varepsilon\phi_{\mu\nu} + O(\varepsilon^2), \chi_{\mu\nu} \approx \chi\phi_{\mu\nu} + O(\varepsilon^2)$$
$$g_{\mu\nu} = \tilde{g}_{\mu\nu} + 2\varepsilon\phi_{\mu\nu} + O(\varepsilon^2),$$
$$g^{\mu\nu} = \tilde{g}^{\mu\nu} + 2\varepsilon\phi^{\mu\nu} + O(\varepsilon^2)$$

So,there are：

$$F_{\mu\nu} \approx \varepsilon\left(\tilde{D}_\mu A_\nu - \tilde{D}_\nu A_\mu\right) + O(\varepsilon^2)$$
$$N^\rho_{\mu\nu} \approx \varepsilon\tilde{g}^{\rho\sigma}\left(\tilde{D}_\mu\phi_{\sigma\nu} + \tilde{D}_\nu\phi_{\sigma\mu} - \tilde{D}_\sigma\phi_{\mu\nu}\right) + O(\varepsilon^2)$$
$$\hat{R}^\rho_{\sigma\mu\nu} \approx \left(\tilde{D}_\mu N^\rho_{\sigma\nu} - \tilde{D}_\nu N^\rho_{\sigma\mu}\right) + O(\varepsilon^2) = \varepsilon\tilde{g}^{\rho m}\begin{pmatrix} \tilde{D}_\mu\tilde{D}_\sigma\phi_{m\nu} - \tilde{D}_\mu\tilde{D}_m\phi_{\sigma\nu} \\ -\tilde{D}_\nu\tilde{D}_\sigma\phi_{m\mu} + \tilde{D}_\nu\tilde{D}_m\phi_{\sigma\mu} \end{pmatrix} + O(\varepsilon^2)$$
$$\hat{R}_{\sigma\nu} \approx \varepsilon\begin{pmatrix} \tilde{D}^\rho\tilde{D}_\sigma\phi_{\rho\nu} - \tilde{D}^\rho\tilde{D}_\rho\phi_{\sigma\nu} \\ -\tilde{D}_\nu\tilde{D}_\sigma\phi + \tilde{D}_\nu\tilde{D}^\rho\phi_{\sigma\rho} \end{pmatrix} + O(\varepsilon^2) = \varepsilon\begin{pmatrix} \tilde{D}_\sigma\tilde{D}^\rho\phi_{\nu\rho} - \tilde{D}^\rho\tilde{D}_\rho\phi_{\sigma\nu} \\ -\tilde{D}_\sigma\tilde{D}_\nu\phi + \tilde{D}_\nu\tilde{D}^\rho\phi_{\sigma\rho} \end{pmatrix} + O(\varepsilon^2)$$
$$\hat{R} \approx 2\varepsilon\left(\tilde{D}^m\tilde{D}^\rho\phi_{m\rho} - \tilde{D}^\rho\tilde{D}_\rho\phi\right) + O(\varepsilon^2)$$
$$\hat{R}_{\sigma\nu} - \frac{1}{2}g_{\sigma\nu}\hat{R} \approx \varepsilon\begin{pmatrix} \tilde{D}_\sigma\tilde{D}^\rho\left(\phi_{\nu\rho} - \frac{1}{2}\tilde{g}_{\nu\rho}\phi\right) - \tilde{D}^\rho\tilde{D}_\rho\left(\phi_{\sigma\nu} - \frac{1}{2}\tilde{g}_{\sigma\nu}\phi\right) \\ +\tilde{D}_\nu\tilde{D}^\rho\left(\phi_{\sigma\rho} - \frac{1}{2}\tilde{g}_{\sigma\rho}\phi\right) - \tilde{g}_{\sigma\nu}\tilde{D}^m\tilde{D}^\rho\left(\phi_{m\rho} - \frac{1}{2}\tilde{g}_{m\rho}\phi\right) \end{pmatrix} + O(\varepsilon^2)$$

$$\kappa_1\hat{R}^{\rho\sigma\mu\nu} = \varepsilon\kappa_1\left(\tilde{D}^\mu M^{\rho\sigma\nu} - \tilde{D}^\nu M^{\rho\sigma\mu}\right) + O(\varepsilon^2)$$
$$\kappa_2\hat{R}^{\mu\nu\rho\sigma} = \varepsilon\kappa_2\left(\tilde{D}^\rho M^{\mu\nu\sigma} - \tilde{D}^\sigma M^{\mu\nu\rho}\right) + O(\varepsilon^2)$$

$$\kappa_3\left(\hat{R}^{\rho\mu\sigma\nu} - \hat{R}^{\rho\nu\sigma\mu} - \hat{R}^{\sigma\mu\rho\nu} + \hat{R}^{\sigma\nu\rho\mu}\right)$$
$$= \varepsilon\kappa_3\begin{pmatrix} \left(\tilde{D}^\sigma M^{\rho\mu\nu} - \tilde{D}^\nu M^{\rho\mu\sigma}\right) \\ -\left(\tilde{D}^\sigma M^{\rho\nu\mu} - \tilde{D}^\mu M^{\rho\nu\sigma}\right) \\ -\left(\tilde{D}^\rho M^{\sigma\mu\nu} - \tilde{D}^\nu M^{\sigma\mu\rho}\right) \\ +\left(\tilde{D}^\rho M^{\sigma\nu\mu} - \tilde{D}^\mu M^{\sigma\nu\rho}\right) \end{pmatrix} + O(\varepsilon^2)$$
$$= \varepsilon\kappa_3\begin{pmatrix} +\tilde{D}^\mu\left(M^{\rho\nu\sigma} - M^{\sigma\nu\rho}\right) - \tilde{D}^\nu\left(M^{\rho\mu\sigma} - M^{\sigma\mu\rho}\right) \\ +\tilde{D}^\rho\left(M^{\mu\sigma\nu} - M^{\nu\sigma\mu}\right) - \tilde{D}^\sigma\left(M^{\mu\rho\nu} - M^{\nu\rho\mu}\right) \end{pmatrix} + O(\varepsilon^2)$$
$$= \varepsilon\kappa_3\begin{pmatrix} +\tilde{D}^\mu\left(M^{\rho\nu\sigma} - M^{\sigma\nu\rho}\right) \\ -\tilde{D}^\nu\left(M^{\rho\mu\sigma} - M^{\sigma\mu\rho}\right) \end{pmatrix} + \varepsilon\kappa_3\begin{pmatrix} +\tilde{D}^\rho\left(M^{\mu\sigma\nu} - M^{\nu\sigma\mu}\right) \\ -\tilde{D}^\sigma\left(M^{\mu\rho\nu} - M^{\nu\rho\mu}\right) \end{pmatrix} + O(\varepsilon^2)$$
$$= \varepsilon\kappa_3\begin{pmatrix} -\tilde{D}^\mu\left(M^{\rho\sigma\nu} - \left(M^{\rho\nu\sigma} - M^{\sigma\nu\rho}\right)\right) \\ +\tilde{D}^\nu\left(M^{\rho\sigma\mu} - \left(M^{\rho\mu\sigma} - M^{\sigma\mu\rho}\right)\right) \\ -\tilde{D}^\rho\left(M^{\mu\nu\sigma} - \left(M^{\mu\sigma\nu} - M^{\nu\sigma\mu}\right)\right) \\ +\tilde{D}^\sigma\left(M^{\mu\nu\rho} - \left(M^{\mu\rho\nu} - M^{\nu\rho\mu}\right)\right) \end{pmatrix} + \varepsilon\kappa_3\begin{pmatrix} \tilde{D}^\mu M^{\rho\sigma\nu} - \tilde{D}^\nu M^{\rho\sigma\mu} \\ +\tilde{D}^\rho M^{\mu\nu\sigma} - \tilde{D}^\sigma M^{\mu\nu\rho} \end{pmatrix} + O(\varepsilon^2)$$

$$\hat{\hat{R}}_{\sigma\nu} = g^{\rho\mu}\hat{\hat{R}}_{\rho\sigma\mu\nu} \approx \varepsilon\left(\tilde{D}^{\rho}M_{\rho\sigma\nu} + \tilde{D}_{\nu}M_{\sigma}\right) + O(\varepsilon^2)$$

$$\kappa_4\left(g^{\rho\mu}\hat{R}^{\nu\sigma} - g^{\rho\nu}\hat{R}^{\mu\sigma} - g^{\sigma\mu}\hat{R}^{\nu\rho} + g^{\sigma\nu}\hat{R}^{\mu\rho}\right)$$

$$= \varepsilon\kappa_4\begin{pmatrix} g^{\rho\mu}\left(\tilde{D}_m M^{m\sigma\nu} + \tilde{D}^{\nu}M^{\sigma}\right) \\ -g^{\rho\nu}\left(\tilde{D}_m M^{m\sigma\mu} + \tilde{D}^{\mu}M^{\sigma}\right) \\ -g^{\sigma\mu}\left(\tilde{D}_m M^{m\rho\nu} + \tilde{D}^{\nu}M^{\rho}\right) \\ +g^{\sigma\nu}\left(\tilde{D}_m M^{m\rho\mu} + \tilde{D}^{\mu}M^{\rho}\right) \end{pmatrix} + O(\varepsilon^2)$$

$$= -\varepsilon\kappa_4\begin{pmatrix} g^{\rho\mu}\tilde{D}_m M^{\sigma m\nu} - g^{\rho\nu}\tilde{D}_m M^{\sigma m\mu} \\ -g^{\sigma\mu}\tilde{D}_m M^{\rho m\nu} + g^{\sigma\nu}\tilde{D}_m M^{\rho m\mu} \end{pmatrix} + \varepsilon\kappa_4\begin{pmatrix} g^{\rho\mu}\tilde{D}^{\nu}M^{\sigma} - g^{\rho\nu}\tilde{D}^{\mu}M^{\sigma} \\ -g^{\sigma\mu}\tilde{D}^{\nu}M^{\rho} + g^{\sigma\nu}\tilde{D}^{\mu}M^{\rho} \end{pmatrix} + O(\varepsilon^2)$$

$$\kappa_5\left(g^{\rho\mu}\hat{R}^{\sigma\nu} - g^{\rho\nu}\hat{R}^{\sigma\mu} - g^{\sigma\mu}\hat{R}^{\rho\nu} + g^{\sigma\nu}\hat{R}^{\rho\mu}\right)$$

$$= \varepsilon\kappa_5\begin{pmatrix} g^{\rho\mu}\left(\tilde{D}_m M^{m\nu\sigma} + \tilde{D}^{\sigma}M^{\nu}\right) \\ -g^{\rho\nu}\left(\tilde{D}_m M^{m\mu\sigma} + \tilde{D}^{\sigma}M^{\mu}\right) \\ -g^{\sigma\mu}\left(\tilde{D}_m M^{m\nu\rho} + \tilde{D}^{\rho}M^{\nu}\right) \\ +g^{\sigma\nu}\left(\tilde{D}_m M^{m\mu\rho} + \tilde{D}^{\rho}M^{\mu}\right) \end{pmatrix} + O(\varepsilon^2)$$

$$= -\varepsilon\kappa_5\begin{pmatrix} g^{\rho\mu}\tilde{D}_m M^{\nu m\sigma} - g^{\rho\nu}\tilde{D}_m M^{\mu m\sigma} \\ -g^{\sigma\mu}\tilde{D}_m M^{\nu m\rho} + g^{\sigma\nu}\tilde{D}_m M^{\mu m\rho} \end{pmatrix} + \varepsilon\kappa_5\begin{pmatrix} g^{\rho\mu}\tilde{D}^{\sigma}M^{\nu} - g^{\rho\nu}\tilde{D}^{\sigma}M^{\mu} \\ -g^{\sigma\mu}\tilde{D}^{\rho}M^{\nu} + g^{\sigma\nu}\tilde{D}^{\rho}M^{\mu} \end{pmatrix} + O(\varepsilon^2)$$

$$\hat{\hat{R}} = g^{\sigma\nu}\hat{\hat{R}}_{\sigma\nu} \approx 2\varepsilon\tilde{D}^{\rho}M_{\rho} + O(\varepsilon^2)$$

$$\kappa_6\left(g^{\rho\mu}g^{\sigma\nu} - g^{\rho\nu}g^{\sigma\mu}\right)\hat{R}$$

$$= 2\kappa_6\left(g^{\rho\mu}g^{\sigma\nu} - g^{\rho\nu}g^{\sigma\mu}\right)\tilde{D}_m M^m$$

$$\hat{\hat{R}}_{\sigma\nu} - \frac{1}{2}g_{\sigma\nu}\hat{\hat{R}} = \varepsilon\tilde{D}^{\rho}\left(M_{\rho\sigma\nu} + \left(\tilde{g}_{\rho\nu}M_{\sigma} - \tilde{g}_{\sigma\nu}M_{\rho}\right)\right) + O(\varepsilon^2)$$

$$\hat{F}_{\lambda\mu\nu} \approx \varepsilon\tilde{F}_{\lambda\mu\nu} + O(\varepsilon^2), \hat{H}_{\lambda\mu\nu} \approx \varepsilon\tilde{H}_{\lambda\mu\nu} + O(\varepsilon^2)$$

$$\hat{Q}_{\lambda\mu\nu} \approx \varepsilon\tilde{Q}_{\lambda\mu\nu} + O(\varepsilon^2), \hat{K}_{\lambda\mu\nu} \approx \varepsilon\tilde{K}_{\lambda\mu\nu} + O(\varepsilon^2)$$

$$\tilde{F}_{\lambda\mu\nu} = \tilde{D}_{\mu}h_{\nu\lambda} - \tilde{D}_{\nu}h_{\mu\lambda} = \left(\tilde{D}_{\mu}\phi_{\nu\lambda} - \tilde{D}_{\nu}\phi_{\mu\lambda}\right) + \left(\tilde{D}_{\mu}\chi_{\nu\lambda} - \tilde{D}_{\nu}\chi_{\mu\lambda}\right)$$

$$\tilde{F}_{\nu} = \left(\tilde{D}^m\phi_{\nu m} - \tilde{D}_{\nu}\phi\right) + \tilde{D}^m\chi_{\nu m}$$

$$\left(\hat{F}_{\mu(\alpha\beta)} - \lambda_{(\alpha)\mu}\hat{F}_{(\beta)} + \lambda_{(\beta)\mu}\hat{F}_{(\alpha)}\right) = \tilde{\lambda}^{\rho}_{(\alpha)}\tilde{\lambda}^{\sigma}_{(\beta)}\varepsilon\left(\tilde{F}_{\mu\rho\sigma} - \tilde{g}_{\rho\mu}\tilde{F}_{\sigma} + \tilde{g}_{\sigma\mu}\tilde{F}_{\rho}\right)$$

$$\left(\tilde{F}_{\mu\rho\sigma} - \tilde{g}_{\rho\mu}\tilde{F}_{\sigma} + \tilde{g}_{\sigma\mu}\tilde{F}_{\rho}\right) = \begin{pmatrix} \begin{pmatrix} \tilde{D}_{\rho}\left(\phi_{\sigma\mu} - \frac{1}{2}\tilde{g}_{\sigma\mu}\phi\right) - \tilde{D}_{\sigma}\left(\phi_{\rho\mu} - \frac{1}{2}\tilde{g}_{\rho\mu}\phi\right) \\ -\tilde{g}_{\rho\mu}\tilde{D}^m\left(\phi_{\sigma m} - \frac{1}{2}\tilde{g}_{\sigma m}\phi\right) + \tilde{g}_{\sigma\mu}\tilde{D}^m\left(\phi_{\rho m} - \frac{1}{2}\tilde{g}_{\rho m}\phi\right) \end{pmatrix} \\ +\left(\tilde{D}_{\rho}\chi_{\sigma\mu} - \tilde{D}_{\sigma}\chi_{\rho\mu}\right) + \left(\tilde{g}_{\rho\mu}\tilde{D}^m\chi_{\sigma m} - \tilde{g}_{\sigma\mu}\tilde{D}^m\chi_{\rho m}\right) \end{pmatrix}$$

$$\tilde{K}_{\mu\nu\lambda} = \tilde{K}_{\lambda\mu\nu} = \tilde{F}_{\lambda\mu\nu} - \tilde{F}_{\mu\lambda\nu} + \tilde{F}_{\nu\lambda\mu} = 2\left(\tilde{D}_{\lambda}\chi_{\mu\nu} - \tilde{D}_{\mu}\chi_{\lambda\nu} + \tilde{D}_{\nu}\chi_{\lambda\mu}\right)$$

$$\hat{F}_{\mu\nu} \approx \varepsilon\tilde{F}_{\mu\nu} + O(\varepsilon^2), \tilde{F}_{\mu\nu} = \tilde{D}_{\mu}A_{\nu} - \tilde{D}_{\nu}A_{\mu}$$

$$P_e^{(\alpha)\mu} \approx \varepsilon\tilde{\lambda}^{(\alpha)}_{\rho}\tilde{P}_e^{\rho\mu} + O(\varepsilon^2), \tilde{P}_e^{\rho\mu} = \frac{1}{2}c\left(i\hbar\bar{\psi}\tilde{\gamma}^{\mu}\tilde{D}^{\rho}\psi - i\hbar\tilde{D}^{\rho}\bar{\psi}\tilde{\gamma}^{\mu}\psi\right)$$

$$P_f^{(\alpha)\mu} \approx O(\varepsilon^2), P_{\gamma}^{(\alpha)\mu} \approx O(\varepsilon^2), P_g^{(\alpha)\mu} \approx O(\varepsilon^2), P_{gk}^{(\alpha)\mu} \approx O(\varepsilon^2)$$

$$\gamma^{\mu} \approx \tilde{\gamma}^{\mu} + O(\varepsilon), \tilde{\gamma}^{\mu} \approx \tilde{\lambda}^{\mu}_{(\alpha)}\gamma^{(\alpha)}, \tilde{\sigma}^{(\alpha\beta)} = \frac{i}{4}\left(\gamma^{(\alpha)}\gamma^{(\beta)} - \gamma^{(\beta)}\gamma^{(\alpha)}\right)$$

$$j_e^{\mu} = \varepsilon\tilde{j}_e^{\mu} + O(\varepsilon^2), \tilde{j}_e^{\mu} = e\bar{\psi}\tilde{\gamma}^{\mu}\psi$$

$$s_e^{(\alpha\beta)\mu} = \varepsilon\tilde{\lambda}_\rho^{(\alpha)}\tilde{\lambda}_\sigma^{(\beta)}\tilde{s}_e^{\rho\sigma\mu}, \tilde{s}_e^{\rho\sigma\mu} = \frac{ch}{4}\bar{\psi}\left(\tilde{\sigma}^{\rho\sigma}\tilde{\gamma}^\mu + \tilde{\gamma}^\mu\tilde{\sigma}^{\rho\sigma}\right)\psi$$

Linear Approximation of the Field Equation of Motion

Substituting the linear approximation of the above field physical quantities into the field equations of motion in the flat background space-time, we obtain the linear approximation equations (only the linear terms are retained):

（1）Dirac electron field equation of motion:

$$\begin{cases} \frac{1}{2}\left(i\hbar\gamma^\mu\tilde{D}_\mu\psi + i\hbar\tilde{D}_\mu(\gamma^\mu\psi)\right) - mc\psi = 0 \\ \frac{1}{2}\left(i\hbar\tilde{D}_\mu\bar{\psi}\gamma^\mu + i\hbar\tilde{D}_\mu(\bar{\psi}\gamma^\mu)\right) + mc\bar{\psi} = 0 \end{cases}$$

……(9.1)

（2）Maxwell's equation of electromagnetic field motion:

$$\tilde{D}_\mu\tilde{D}^\nu A_\nu - \tilde{D}_\nu\tilde{D}^\nu A_\mu = 4\pi\tilde{j}_e^\mu$$

……(9.2)

（3）Equation of motion of spin field:

When ($\kappa_4 = -\kappa_5$), we get:

$$\begin{pmatrix} \left(\kappa_1+\kappa_3\right)\tilde{D}_\nu\left(\tilde{D}^\mu M^{\rho\sigma\nu} - \tilde{D}^\nu M^{\rho\sigma\mu}\right) \\ +\left(\kappa_2+\kappa_3\right)\tilde{D}_\nu\left(\tilde{D}^\rho M^{\mu\nu\sigma} - \tilde{D}^\sigma M^{\mu\nu\rho}\right) \\ +\kappa_4\tilde{D}_\nu\begin{pmatrix} g^{\rho\mu}\tilde{D}_m M^{\sigma\nu m} - g^{\rho\nu}\tilde{D}_m M^{\sigma\mu m} \\ -g^{\sigma\mu}\tilde{D}_m M^{\rho\nu m} + g^{\sigma\nu}\tilde{D}_m M^{\rho\mu m} \end{pmatrix} \\ +\kappa_3\tilde{D}_\nu\begin{pmatrix} -\tilde{D}^\mu\left(M^{\rho\sigma\nu} - \left(M^{\rho\nu\sigma} - M^{\sigma\nu\rho}\right)\right) \\ +\tilde{D}^\nu\left(M^{\rho\sigma\mu} - \left(M^{\rho\mu\sigma} - M^{\sigma\mu\rho}\right)\right) \\ -\tilde{D}^\rho\left(M^{\mu\nu\sigma} - \left(M^{\mu\sigma\nu} - M^{\nu\sigma\mu}\right)\right) \\ +\tilde{D}^\sigma\left(M^{\mu\nu\rho} - \left(M^{\mu\rho\nu} - M^{\nu\rho\mu}\right)\right) \end{pmatrix} \\ -\kappa_4\begin{pmatrix} -\tilde{D}^\rho\tilde{D}_m\left(M^{\sigma\mu m} - \left(M^{\sigma m\mu} - M^{\mu m\sigma}\right)\right) \\ +\tilde{D}^\sigma\tilde{D}_m\left(M^{\rho\mu m} - \left(M^{\rho m\mu} - M^{\mu m\rho}\right)\right) \end{pmatrix} \\ +\kappa_4\tilde{D}_\nu\begin{pmatrix} \tilde{g}^{\rho\mu}\left(\tilde{D}^\nu M^\sigma - \tilde{D}^\sigma M^\nu\right) \\ -\tilde{g}^{\rho\nu}\left(\tilde{D}^\mu M^\sigma - \tilde{D}^\sigma M^\mu\right) \\ -\tilde{g}^{\sigma\mu}\left(\tilde{D}^\nu M^\rho - \tilde{D}^\rho M^\nu\right) \\ +\tilde{g}^{\sigma\nu}\left(\tilde{D}^\mu M^\rho - \tilde{D}^\rho M^\mu\right) \end{pmatrix} \\ +2\kappa_6\left(\tilde{g}^{\rho\mu}\tilde{D}^\sigma\tilde{D}_m M^m - \tilde{g}^{\sigma\mu}\tilde{D}^\rho\tilde{D}_m M^m\right) \\ -\frac{\kappa}{G_N}\alpha\left(M^{\rho\sigma\mu} + \left(\tilde{g}^{\rho\mu}M^\sigma - \tilde{g}^{\sigma\mu}M^\rho\right)\right) \\ +\frac{\kappa}{G_N}\left(2\beta+\alpha\right)\left(M^{\rho\sigma\mu} - \left(M^{\rho\mu\sigma} - M^{\sigma\mu\rho}\right)\right) \end{pmatrix} = \begin{pmatrix} \frac{8\pi\kappa}{c^4}\tilde{s}_e^{\rho\sigma\mu} - \alpha\frac{\kappa}{G_N}\left(\tilde{D}^\mu\chi^{\rho\sigma} - \tilde{g}^{\rho\mu}\tilde{D}_m\chi^{\sigma m} + g^{\sigma\mu}\tilde{D}_m\chi^{\rho m}\right) \\ +\left(2\beta+\alpha\right)\frac{\kappa}{G_N}\left(\tilde{D}^\mu\chi^{\rho\sigma} - \left(\tilde{D}^\rho\chi^{\mu\sigma} - \tilde{D}^\sigma\chi^{\mu\rho}\right)\right) \\ +\alpha\frac{\kappa}{G_N}\begin{pmatrix} \tilde{D}^\rho\left(\phi^{\sigma\mu} - \frac{1}{2}\tilde{g}^{\sigma\mu}\phi\right) - \tilde{D}^\sigma\left(\phi^{\rho\mu} - \frac{1}{2}\tilde{g}^{\rho\mu}\phi\right) \\ -\tilde{g}^{\rho\mu}\tilde{D}_m\left(\phi^{\sigma m} - \frac{1}{2}\tilde{g}^{\sigma m}\phi\right) \\ +\tilde{g}^{\sigma\mu}\tilde{D}_m\left(\phi^{\rho m} - \frac{1}{2}\tilde{g}^{\to\rho m}\phi\right) \end{pmatrix} \end{pmatrix}$$

When $\kappa_1 = 1, \kappa_2 = -\kappa_3$, the above equation becomes:

$$
\begin{pmatrix}
(1-\kappa_2)\tilde{D}_\nu\left(\tilde{D}^\mu M^{\rho\sigma\nu}-\tilde{D}^\nu M^{\rho\sigma\mu}\right)\\
+\kappa_4\tilde{D}_\nu\begin{pmatrix} g^{\rho\mu}\tilde{D}_m M^{\sigma\nu m}-g^{\rho\nu}\tilde{D}_m M^{\sigma\mu m}\\ -g^{\sigma\mu}\tilde{D}_m M^{\rho\nu m}+g^{\sigma\nu}\tilde{D}_m M^{\rho\mu m}\end{pmatrix}\\
-\kappa_2\tilde{D}_\nu\begin{pmatrix} -\tilde{D}^\mu\left(M^{\rho\sigma\nu}-\left(M^{\rho\nu\sigma}-M^{\sigma\nu\rho}\right)\right)\\ +\tilde{D}^\nu\left(M^{\rho\sigma\mu}-\left(M^{\rho\mu\sigma}-M^{\sigma\mu\rho}\right)\right)\\ -\tilde{D}^\rho\left(M^{\mu\nu\sigma}-\left(M^{\mu\sigma\nu}-M^{\nu\sigma\mu}\right)\right)\\ +\tilde{D}^\sigma\left(M^{\mu\nu\rho}-\left(M^{\mu\rho\nu}-M^{\nu\rho\mu}\right)\right)\end{pmatrix}\\
-\kappa_4\begin{pmatrix} -\tilde{D}^\rho\tilde{D}_m\left(M^{\sigma\mu m}-\left(M^{\sigma m\mu}-M^{\mu m\sigma}\right)\right)\\ +\tilde{D}^\sigma\tilde{D}_m\left(M^{\rho\mu m}-\left(M^{\rho m\mu}-M^{\mu m\rho}\right)\right)\end{pmatrix}\\
+\kappa_4\tilde{D}_\nu\begin{pmatrix} \tilde{g}^{\rho\mu}\left(\tilde{D}^\nu M^\sigma-\tilde{D}^\sigma M^\nu\right)\\ -\tilde{g}^{\rho\nu}\left(\tilde{D}^\mu M^\sigma-\tilde{D}^\sigma M^\mu\right)\\ -\tilde{g}^{\sigma\mu}\left(\tilde{D}^\nu M^\rho-\tilde{D}^\rho M^\nu\right)\\ +\tilde{g}^{\sigma\nu}\left(\tilde{D}^\mu M^\rho-\tilde{D}^\rho M^\mu\right)\end{pmatrix}\\
+2\kappa_6\left(\tilde{g}^{\rho\mu}\tilde{D}^\sigma\tilde{D}_m M^m-\tilde{g}^{\sigma\mu}\tilde{D}^\rho\tilde{D}_m M^m\right)\\
-\frac{\kappa}{G_N}\alpha\left(M^{\rho\sigma\mu}+\left(\tilde{g}^{\rho\mu}M^\sigma-\tilde{g}^{\sigma\mu}M^\rho\right)\right)\\
+\frac{\kappa}{G_N}(2\beta+\alpha)\left(M^{\rho\sigma\mu}-\left(M^{\rho\mu\sigma}-M^{\sigma\mu\rho}\right)\right)
\end{pmatrix}
=
\begin{pmatrix}
\frac{8\pi\kappa}{c^4}\tilde{s}_e^{\rho\sigma\mu}-\alpha\frac{\kappa}{G_N}\left(\tilde{D}^\mu\chi^{\rho\sigma}-\tilde{g}^{\rho\mu}\tilde{D}_m\chi^{\sigma m}+g^{\sigma\mu}\tilde{D}_m\chi^{\rho m}\right)\\
+(2\beta+\alpha)\frac{\kappa}{G_N}\left(\tilde{D}^\mu\chi^{\rho\sigma}-\left(\tilde{D}^\rho\chi^{\mu\sigma}-\tilde{D}^\sigma\chi^{\mu\rho}\right)\right)\\
+\alpha\frac{\kappa}{G_N}\begin{pmatrix} \tilde{D}^\rho\left(\phi^{\sigma\mu}-\frac{1}{2}\tilde{g}^{\sigma\mu}\phi\right)-\tilde{D}^\sigma\left(\phi^{\rho\mu}-\frac{1}{2}\tilde{g}^{\rho\mu}\phi\right)\\ -\tilde{g}^{\rho\mu}\tilde{D}_m\left(\phi^{\sigma m}-\frac{1}{2}\tilde{g}^{\sigma m}\phi\right)\\ +\tilde{g}^{\sigma\mu}\tilde{D}_m\left(\phi^{\rho m}-\frac{1}{2}\tilde{g}^{\to\rho m}\phi\right)\end{pmatrix}
\end{pmatrix}
$$

$$
\left(\begin{array}{l}
(1-\kappa_2)\tilde{D}_\nu\begin{pmatrix} \tilde{D}^\mu\left(M^{\rho\sigma\nu}+\left(\tilde{g}^{\rho\nu}M^\sigma-\tilde{g}^{\sigma\nu}M^\rho\right)\right)\\ -\tilde{D}^\nu\left(M^{\rho\sigma\mu}+\left(\tilde{g}^{\rho\mu}M^\sigma-\tilde{g}^{\sigma\mu}M^\rho\right)\right)\end{pmatrix}\\
-(1-\kappa_2)\tilde{D}_\nu\begin{pmatrix} \tilde{D}^\mu\left(\tilde{g}^{\rho\nu}M^\sigma-\tilde{g}^{\sigma\nu}M^\rho\right)\\ -\tilde{D}^\nu\left(\tilde{g}^{\rho\mu}M^\sigma-\tilde{g}^{\sigma\mu}M^\rho\right)\end{pmatrix}\\
+\kappa_4\tilde{D}_\nu\begin{pmatrix} g^{\rho\mu}\tilde{D}_m\left(M^{\sigma\nu m}+\left(\tilde{g}^{\sigma m}M^\nu-\tilde{g}^{\nu m}M^\sigma\right)\right)\\ -g^{\rho\nu}\tilde{D}_m\left(M^{\sigma\mu m}+\left(\tilde{g}^{\sigma m}M^\mu-\tilde{g}^{\mu m}M^\sigma\right)\right)\\ -g^{\sigma\mu}\tilde{D}_m\left(M^{\rho\nu m}+\left(\tilde{g}^{\rho m}M^\nu-\tilde{g}^{\nu m}M^\rho\right)\right)\\ +g^{\sigma\nu}\tilde{D}_m\left(M^{\rho\mu m}+\left(\tilde{g}^{\rho m}M^\mu-\tilde{g}^{\mu m}M^\rho\right)\right)\end{pmatrix}\\
-\kappa_4\tilde{D}_\nu\begin{pmatrix} g^{\rho\mu}\tilde{D}_m\left(\tilde{g}^{\sigma m}M^\nu-\tilde{g}^{\nu m}M^\sigma\right)\\ -g^{\rho\nu}\tilde{D}_m\left(\tilde{g}^{\sigma m}M^\mu-\tilde{g}^{\mu m}M^\sigma\right)\\ -g^{\sigma\mu}\tilde{D}_m\left(\tilde{g}^{\rho m}M^\nu-\tilde{g}^{\nu m}M^\rho\right)\\ +g^{\sigma\nu}\tilde{D}_m\left(\tilde{g}^{\rho m}M^\mu-\tilde{g}^{\mu m}M^\rho\right)\end{pmatrix}\\
-\kappa_2\tilde{D}_\nu\begin{pmatrix} -\tilde{D}^\mu\left(M^{\rho\sigma\nu}-\left(M^{\rho\nu\sigma}-M^{\sigma\nu\rho}\right)\right)\\ +\tilde{D}^\nu\left(M^{\rho\sigma\mu}-\left(M^{\rho\mu\sigma}-M^{\sigma\mu\rho}\right)\right)\\ +\tilde{D}^\rho\left(M^{\mu\sigma\nu}-\left(M^{\mu\nu\sigma}-M^{\sigma\nu\mu}\right)\right)\\ -\tilde{D}^\sigma\left(M^{\mu\rho\nu}-\left(M^{\mu\nu\rho}-M^{\rho\nu\mu}\right)\right)\end{pmatrix}\\
-\kappa_4\begin{pmatrix} -\tilde{D}^\rho\tilde{D}_m\left(M^{\sigma\mu m}-\left(M^{\sigma m\mu}-M^{\mu m\sigma}\right)\right)\\ +\tilde{D}^\sigma\tilde{D}_m\left(M^{\rho\mu m}-\left(M^{\rho m\mu}-M^{\mu m\rho}\right)\right)\end{pmatrix}
\end{array}\right.
=
\begin{pmatrix}
\frac{8\pi\kappa}{c^4}\tilde{s}_e^{\rho\sigma\mu}-\alpha\frac{\kappa}{G_N}\left(\tilde{D}^\mu\chi^{\rho\sigma}-\tilde{g}^{\rho\mu}\tilde{D}_m\chi^{\sigma m}+g^{\sigma\mu}\tilde{D}_m\chi^{\rho m}\right)\\
+(2\beta+\alpha)\frac{\kappa}{G_N}\left(\tilde{D}^\mu\chi^{\rho\sigma}-\left(\tilde{D}^\rho\chi^{\mu\sigma}-\tilde{D}^\sigma\chi^{\mu\rho}\right)\right)\\
+\alpha\frac{\kappa}{G_N}\begin{pmatrix} \tilde{D}^\rho\left(\phi^{\sigma\mu}-\frac{1}{2}\tilde{g}^{\sigma\mu}\phi\right)-\tilde{D}^\sigma\left(\phi^{\rho\mu}-\frac{1}{2}\tilde{g}^{\rho\mu}\phi\right)\\ -\tilde{g}^{\rho\mu}\tilde{D}_m\left(\phi^{\sigma m}-\frac{1}{2}\tilde{g}^{\sigma m}\phi\right)\\ +\tilde{g}^{\sigma\mu}\tilde{D}_m\left(\phi^{\rho m}-\frac{1}{2}\tilde{g}^{\to\rho m}\phi\right)\end{pmatrix}
\end{pmatrix}
$$

$$
\left.
\begin{array}{l}
+\kappa_4 \tilde{D}_\nu \left(\begin{array}{l} \tilde{g}^{\rho\mu}\left(\tilde{D}^\nu M^\sigma - \tilde{D}^\sigma M^\nu\right) \\ -\tilde{g}^{\rho\nu}\left(\tilde{D}^\mu M^\sigma - \tilde{D}^\sigma M^\mu\right) \\ -\tilde{g}^{\sigma\mu}\left(\tilde{D}^\nu M^\rho - \tilde{D}^\rho M^\nu\right) \\ +\tilde{g}^{\sigma\nu}\left(\tilde{D}^\mu M^\rho - \tilde{D}^\rho M^\mu\right) \end{array} \right) \\
+2\kappa_6\left(\tilde{g}^{\rho\mu}\tilde{D}^\sigma \tilde{D}_m M^m - \tilde{g}^{\sigma\mu}\tilde{D}^\rho \tilde{D}_m M^m\right) \\
-\frac{\kappa}{G_N}\alpha\left(M^{\rho\sigma\mu} + \left(\tilde{g}^{\rho\mu}M^\sigma - \tilde{g}^{\sigma\mu}M^\rho\right)\right) \\
+\frac{\kappa}{G_N}(2\beta+\alpha)\left(M^{\rho\sigma\mu} - \left(M^{\rho\mu\sigma} - M^{\sigma\mu\rho}\right)\right)
\end{array}
\right)
$$

$$
\left(
\begin{array}{l}
(1-\kappa_2)\left(\begin{array}{l} \tilde{D}^\mu \tilde{D}_\nu\left(M^{\rho\sigma\nu} + \left(\tilde{g}^{\rho\nu}M^\sigma - \tilde{g}^{\sigma\nu}M^\rho\right)\right) \\ -\tilde{D}^\nu \tilde{D}_\nu\left(M^{\rho\sigma\mu} + \left(\tilde{g}^{\rho\mu}M^\sigma - \tilde{g}^{\sigma\mu}M^\rho\right)\right) \end{array} \right) \\
+\kappa_4 \tilde{D}_\nu \left(\begin{array}{l} g^{\rho\mu}\tilde{D}_m\left(M^{\sigma\nu m} + \left(\tilde{g}^{\sigma m}M^\nu - \tilde{g}^{\nu m}M^\sigma\right)\right) \\ -g^{\rho\nu}\tilde{D}_m\left(M^{\sigma\mu m} + \left(\tilde{g}^{\sigma m}M^\mu - \tilde{g}^{\mu m}M^\sigma\right)\right) \\ -g^{\sigma\mu}\tilde{D}_m\left(M^{\rho\nu m} + \left(\tilde{g}^{\rho m}M^\nu - \tilde{g}^{\nu m}M^\rho\right)\right) \\ +g^{\sigma\nu}\tilde{D}_m\left(M^{\rho\mu m} + \left(\tilde{g}^{\rho m}M^\mu - \tilde{g}^{\mu m}M^\rho\right)\right) \end{array} \right) \\
-\kappa_2 \left(\begin{array}{l} -\tilde{D}^\mu \tilde{D}_\nu\left(M^{\rho\sigma\nu} - \left(M^{\rho\nu\sigma} - M^{\sigma\nu\rho}\right)\right) \\ +\tilde{D}^\nu \tilde{D}_\nu\left(M^{\rho\sigma\mu} - \left(M^{\rho\mu\sigma} - M^{\sigma\mu\rho}\right)\right) \\ +\tilde{D}^\rho \tilde{D}_m\left(M^{\mu\sigma m} - \left(M^{\mu m\sigma} - M^{\sigma m\mu}\right)\right) \\ -\tilde{D}^\sigma \tilde{D}_m\left(M^{\mu\rho m} - \left(M^{\mu m\rho} - M^{\rho m\mu}\right)\right) \end{array} \right) \\
-\kappa_4 \left(\begin{array}{l} +\tilde{D}^\rho \tilde{D}_m\left(M^{\mu\sigma m} - \left(M^{\mu m\sigma} - M^{\sigma m\mu}\right)\right) \\ -\tilde{D}^\sigma \tilde{D}_m\left(M^{\mu\rho m} - \left(M^{\mu m\rho} - M^{\rho m\mu}\right)\right) \end{array} \right) \\
-(1-\kappa_2)\tilde{D}_\nu \left(\begin{array}{l} \tilde{D}^\mu\left(\tilde{g}^{\rho\nu}M^\sigma - \tilde{g}^{\sigma\nu}M^\rho\right) \\ -\tilde{D}^\nu\left(\tilde{g}^{\rho\mu}M^\sigma - \tilde{g}^{\sigma\mu}M^\rho\right) \end{array} \right) \\
-\kappa_4 \tilde{D}_\nu \left(\begin{array}{l} g^{\rho\mu}\tilde{D}_m\left(\tilde{g}^{\sigma m}M^\nu - \tilde{g}^{\nu m}M^\sigma\right) \\ -g^{\rho\nu}\tilde{D}_m\left(\tilde{g}^{\sigma m}M^\mu - \tilde{g}^{\mu m}M^\sigma\right) \\ -g^{\sigma\mu}\tilde{D}_m\left(\tilde{g}^{\rho m}M^\nu - \tilde{g}^{\nu m}M^\rho\right) \\ +g^{\sigma\nu}\tilde{D}_m\left(\tilde{g}^{\rho m}M^\mu - \tilde{g}^{\mu m}M^\rho\right) \end{array} \right) \\
+\kappa_4 \tilde{D}_\nu \left(\begin{array}{l} \tilde{g}^{\rho\mu}\left(\tilde{D}^\nu M^\sigma - \tilde{D}^\sigma M^\nu\right) \\ -\tilde{g}^{\rho\nu}\left(\tilde{D}^\mu M^\sigma - \tilde{D}^\sigma M^\mu\right) \\ -\tilde{g}^{\sigma\mu}\left(\tilde{D}^\nu M^\rho - \tilde{D}^\rho M^\nu\right) \\ +\tilde{g}^{\sigma\nu}\left(\tilde{D}^\mu M^\rho - \tilde{D}^\rho M^\mu\right) \end{array} \right) \\
+2\kappa_6\left(\tilde{g}^{\rho\mu}\tilde{D}^\sigma \tilde{D}_m M^m - \tilde{g}^{\sigma\mu}\tilde{D}^\rho \tilde{D}_m M^m\right) \\
-\frac{\kappa}{G_N}\alpha\left(M^{\rho\sigma\mu} + \left(\tilde{g}^{\rho\mu}M^\sigma - \tilde{g}^{\sigma\mu}M^\rho\right)\right) \\
+\frac{\kappa}{G_N}(2\beta+\alpha)\left(M^{\rho\sigma\mu} - \left(M^{\rho\mu\sigma} - M^{\sigma\mu\rho}\right)\right)
\end{array}
\right)
=
\left(
\begin{array}{l}
\frac{8\pi\kappa}{c^4}\tilde{s}_e^{\rho\sigma\mu} - \alpha\frac{\kappa}{G_N}\left(\tilde{D}^\mu \chi^{\rho\sigma} - \tilde{g}^{\rho\mu}\tilde{D}_m \chi^{\sigma m} + g^{\sigma\mu}\tilde{D}_m \chi^{\rho m}\right) \\
+(2\beta+\alpha)\frac{\kappa}{G_N}\left(\tilde{D}^\mu \chi^{\rho\sigma} - \left(\tilde{D}^\rho \chi^{\mu\sigma} - \tilde{D}^\sigma \chi^{\mu\rho}\right)\right) \\
+\alpha\frac{\kappa}{G_N}\left(\begin{array}{l} \tilde{D}^\rho\left(\phi^{\sigma\mu} - \frac{1}{2}\tilde{g}^{\sigma\mu}\phi\right) - \tilde{D}^\sigma\left(\phi^{\rho\mu} - \frac{1}{2}\tilde{g}^{\rho\mu}\phi\right) \\ -\tilde{g}^{\rho\mu}\tilde{D}_m\left(\phi^{\sigma m} - \frac{1}{2}\tilde{g}^{\sigma m}\phi\right) \\ +\tilde{g}^{\sigma\mu}\tilde{D}_m\left(\phi^{\rho m} - \frac{1}{2}\tilde{g}^{\to\rho m}\phi\right) \end{array} \right)
\end{array}
\right)
$$

$$
\left(\begin{array}{l}
(1-\kappa_2)\left(\begin{array}{l}\tilde{D}^{\mu}\tilde{D}_{\nu}\left(M^{\rho\sigma\nu}+\left(\tilde{g}^{\rho\nu}M^{\sigma}-\tilde{g}^{\sigma\nu}M^{\rho}\right)\right)\\ -\tilde{D}^{\nu}\tilde{D}_{\nu}\left(M^{\rho\sigma\mu}+\left(\tilde{g}^{\rho\mu}M^{\sigma}-\tilde{g}^{\sigma\mu}M^{\rho}\right)\right)\end{array}\right)\\
+\kappa_4\tilde{D}_{\nu}\left(\begin{array}{l}g^{\rho\mu}\tilde{D}_m\left(M^{\sigma\nu m}+\left(\tilde{g}^{\sigma m}M^{\nu}-\tilde{g}^{\nu m}M^{\sigma}\right)\right)\\ -g^{\rho\nu}\tilde{D}_m\left(M^{\sigma\mu m}+\left(\tilde{g}^{\sigma m}M^{\mu}-\tilde{g}^{\mu m}M^{\sigma}\right)\right)\\ -g^{\sigma\mu}\tilde{D}_m\left(M^{\rho\nu m}+\left(\tilde{g}^{\rho m}M^{\nu}-\tilde{g}^{\nu m}M^{\rho}\right)\right)\\ +g^{\sigma\nu}\tilde{D}_m\left(M^{\rho\mu m}+\left(\tilde{g}^{\rho m}M^{\mu}-\tilde{g}^{\mu m}M^{\rho}\right)\right)\end{array}\right)\\
-\kappa_2\left(\begin{array}{l}-\tilde{D}^{\mu}\tilde{D}_{\nu}\left(M^{\rho\sigma\nu}-\left(M^{\rho\nu\sigma}-M^{\sigma\nu\rho}\right)\right)\\ +\tilde{D}^{\nu}\tilde{D}_{\nu}\left(M^{\rho\sigma\mu}-\left(M^{\rho\mu\sigma}-M^{\sigma\mu\rho}\right)\right)\end{array}\right)\\
-\left(\kappa_2+\kappa_4\right)\left(\begin{array}{l}+\tilde{D}^{\rho}\tilde{D}_m\left(M^{\mu\sigma m}-\left(M^{\mu m\sigma}-M^{\sigma m\mu}\right)\right)\\ -\tilde{D}^{\sigma}\tilde{D}_m\left(M^{\mu\rho m}-\left(M^{\mu m\rho}-M^{\rho m\mu}\right)\right)\end{array}\right)\\
+\left(2\kappa_4+\left(1-\kappa_2\right)\right)\left(\begin{array}{l}\tilde{D}_m\tilde{D}^m\left(\tilde{g}^{\rho\mu}M^{\sigma}-\tilde{g}^{\sigma\mu}M^{\rho}\right)\\ -\tilde{D}^{\mu}\left(\tilde{D}^{\rho}M^{\sigma}-\tilde{D}^{\sigma}M^{\rho}\right)\end{array}\right)\\
+2\left(\kappa_6-\kappa_4\right)\left(\tilde{g}^{\rho\mu}\tilde{D}^{\sigma}\tilde{D}_mM^m-\tilde{g}^{\sigma\mu}\tilde{D}^{\rho}\tilde{D}_mM^m\right)\\
-\frac{\kappa}{G_N}\alpha\left(M^{\rho\sigma\mu}+\left(\tilde{g}^{\rho\mu}M^{\sigma}-\tilde{g}^{\sigma\mu}M^{\rho}\right)\right)\\
+\frac{\kappa}{G_N}(2\beta+\alpha)\left(M^{\rho\sigma\mu}-\left(M^{\rho\mu\sigma}-M^{\sigma\mu\rho}\right)\right)
\end{array}\right)
=\left(\begin{array}{l}
\frac{8\pi\kappa}{c^4}\tilde{s}_e^{\rho\sigma\mu}-\alpha\frac{\kappa}{G_N}\left(\tilde{D}^{\mu}\chi^{\rho\sigma}-\tilde{g}^{\rho\mu}\tilde{D}_m\chi^{\sigma m}+g^{\sigma\mu}\tilde{D}_m\chi^{\rho m}\right)\\
+\left(2\beta+\alpha\right)\frac{\kappa}{G_N}\left(\tilde{D}^{\mu}\chi^{\rho\sigma}-\left(\tilde{D}^{\rho}\chi^{\mu\sigma}-\tilde{D}^{\sigma}\chi^{\mu\rho}\right)\right)\\
+\alpha\frac{\kappa}{G_N}\left(\begin{array}{l}\tilde{D}^{\rho}\left(\phi^{\sigma\mu}-\frac{1}{2}\tilde{g}^{\sigma\mu}\phi\right)-\tilde{D}^{\sigma}\left(\phi^{\rho\mu}-\frac{1}{2}\tilde{g}^{\rho\mu}\phi\right)\\ -\tilde{g}^{\rho\mu}\tilde{D}_m\left(\phi^{\sigma m}-\frac{1}{2}\tilde{g}^{\sigma m}\phi\right)\\ +\tilde{g}^{\sigma\mu}\tilde{D}_m\left(\phi^{\rho m}-\frac{1}{2}\tilde{g}^{\to\rho m}\phi\right)\end{array}\right)
\end{array}\right)
$$

$$
\left(\begin{array}{l}
(1-\kappa_2)\left(\begin{array}{l}\tilde{D}^{\mu}\tilde{D}_{\nu}\left(M^{\rho\sigma\nu}+\left(\tilde{g}^{\rho\nu}M^{\sigma}-\tilde{g}^{\sigma\nu}M^{\rho}\right)\right)\\ -\tilde{D}^{\nu}\tilde{D}_{\nu}\left(M^{\rho\sigma\mu}+\left(\tilde{g}^{\rho\mu}M^{\sigma}-\tilde{g}^{\sigma\mu}M^{\rho}\right)\right)\end{array}\right)\\
+\kappa_2\left(\begin{array}{l}\tilde{D}^{\mu}\tilde{D}_{\nu}\left(M^{\rho\sigma\nu}-\left(M^{\rho\nu\sigma}-M^{\sigma\nu\rho}\right)\right)\\ -\tilde{D}^{\nu}\tilde{D}_{\nu}\left(M^{\rho\sigma\mu}-\left(M^{\rho\mu\sigma}-M^{\sigma\mu\rho}\right)\right)\end{array}\right)\\
+\kappa_4\tilde{D}_{\nu}\left(\begin{array}{l}\tilde{g}^{\rho\mu}\tilde{D}_m\left(M^{\sigma\nu m}+\left(\tilde{g}^{\sigma m}M^{\nu}-\tilde{g}^{\nu m}M^{\sigma}\right)\right)\\ -\tilde{g}^{\sigma\mu}\tilde{D}_m\left(M^{\rho\nu m}+\left(\tilde{g}^{\rho m}M^{\nu}-\tilde{g}^{\nu m}M^{\rho}\right)\right)\end{array}\right)\\
-\left(\kappa_2+\kappa_4\right)\tilde{D}_{\nu}\left(\begin{array}{l}\tilde{g}^{\rho\mu}\tilde{D}_m\left(M^{\sigma\nu m}-\left(M^{\sigma m\nu}-M^{\nu m\sigma}\right)\right)\\ -\tilde{g}^{\sigma\mu}\tilde{D}_m\left(M^{\rho\nu m}-\left(M^{\rho m\nu}-M^{\nu m\rho}\right)\right)\end{array}\right)\\
-\kappa_4\left(\begin{array}{l}\tilde{D}^{\rho}\tilde{D}_m\left(M^{\sigma\mu m}+\left(\tilde{g}^{\sigma m}M^{\mu}-\tilde{g}^{\mu m}M^{\sigma}\right)\right)\\ -\tilde{D}^{\sigma}\tilde{D}_m\left(M^{\rho\mu m}+\left(\tilde{g}^{\rho m}M^{\mu}-\tilde{g}^{\mu m}M^{\rho}\right)\right)\end{array}\right)\\
+\left(\kappa_2+\kappa_4\right)\left(\begin{array}{l}+\tilde{D}^{\rho}\tilde{D}_m\left(M^{\sigma\mu m}-\left(M^{\sigma m\mu}-M^{\mu m\sigma}\right)\right)\\ -\tilde{D}^{\sigma}\tilde{D}_m\left(M^{\rho\mu m}-\left(M^{\rho m\mu}-M^{\mu m\rho}\right)\right)\end{array}\right)\\
+\left(2\kappa_4+\left(1-\kappa_2\right)\right)\left(\begin{array}{l}\tilde{D}_m\tilde{D}^m\left(\tilde{g}^{\rho\mu}M^{\sigma}-\tilde{g}^{\sigma\mu}M^{\rho}\right)\\ -\tilde{D}^{\mu}\left(\tilde{D}^{\rho}M^{\sigma}-\tilde{D}^{\sigma}M^{\rho}\right)\end{array}\right)
\end{array}\right.
=\left(\begin{array}{l}
\frac{8\pi\kappa}{c^4}\tilde{s}_e^{\rho\sigma\mu}-\alpha\frac{\kappa}{G_N}\left(\tilde{D}^{\mu}\chi^{\rho\sigma}-\tilde{g}^{\rho\mu}\tilde{D}_m\chi^{\sigma m}+g^{\sigma\mu}\tilde{D}_m\chi^{\rho m}\right)\\
+\left(2\beta+\alpha\right)\frac{\kappa}{G_N}\left(\tilde{D}^{\mu}\chi^{\rho\sigma}-\left(\tilde{D}^{\rho}\chi^{\mu\sigma}-\tilde{D}^{\sigma}\chi^{\mu\rho}\right)\right)\\
+\alpha\frac{\kappa}{G_N}\left(\begin{array}{l}\tilde{D}^{\rho}\left(\phi^{\sigma\mu}-\frac{1}{2}\tilde{g}^{\sigma\mu}\phi\right)-\tilde{D}^{\sigma}\left(\phi^{\rho\mu}-\frac{1}{2}\tilde{g}^{\rho\mu}\phi\right)\\ -\tilde{g}^{\rho\mu}\tilde{D}_m\left(\phi^{\sigma m}-\frac{1}{2}\tilde{g}^{\sigma m}\phi\right)\\ +\tilde{g}^{\sigma\mu}\tilde{D}_m\left(\phi^{\rho m}-\frac{1}{2}\tilde{g}^{\to\rho m}\phi\right)\end{array}\right)
\end{array}\right)
$$

$$\begin{pmatrix} +2(\kappa_6-\kappa_4)\left(\tilde{g}^{\rho\mu}\tilde{D}^{\sigma}\tilde{D}_m M^m-\tilde{g}^{\sigma\mu}\tilde{D}^{\rho}\tilde{D}_m M^m\right) \\ -\frac{\kappa}{G_N}\alpha\left(M^{\rho\sigma\mu}+\left(\tilde{g}^{\rho\mu}M^{\sigma}-\tilde{g}^{\sigma\mu}M^{\rho}\right)\right) \\ +\frac{\kappa}{G_N}(2\beta+\alpha)\left(M^{\rho\sigma\mu}-\left(M^{\rho\mu\sigma}-M^{\sigma\mu\rho}\right)\right) \end{pmatrix}$$

A. When $\kappa_1=1,\kappa_2=-\kappa_3=\frac{1}{2},\kappa_4=-\kappa_5=\kappa_6=-\frac{1}{4},\beta=-\alpha$, the above equation has the simplest form:

$$\begin{pmatrix} \begin{pmatrix} \tilde{D}^{\mu}\tilde{D}_{\nu}\left(M^{\rho\sigma\nu}-\frac{1}{2}\left(M^{\rho\nu\sigma}-M^{\sigma\nu\rho}\right)+\frac{1}{2}\left(\tilde{g}^{\rho\nu}M^{\sigma}-\tilde{g}^{\sigma\nu}M^{\rho}\right)\right) \\ -\tilde{D}^{\nu}\tilde{D}_{\nu}\left(M^{\rho\sigma\mu}-\frac{1}{2}\left(M^{\rho\mu\sigma}-M^{\sigma\mu\rho}\right)+\frac{1}{2}\left(\tilde{g}^{\rho\mu}M^{\sigma}-\tilde{g}^{\sigma\mu}M^{\rho}\right)\right) \end{pmatrix} \\ -\frac{1}{2}\tilde{D}_{\nu}\begin{pmatrix} \tilde{g}^{\rho\mu}\tilde{D}_m\left(M^{\sigma\nu m}-\frac{1}{2}\left(M^{\sigma m\nu}-M^{\nu m\sigma}\right)+\frac{1}{2}\left(\tilde{g}^{\sigma m}M^{\nu}-\tilde{g}^{\nu m}M^{\sigma}\right)\right) \\ -\tilde{g}^{\sigma\mu}\tilde{D}_m\left(M^{\rho\nu m}-\frac{1}{2}\left(M^{\rho m\nu}-M^{\nu m\rho}\right)+\frac{1}{2}\left(\tilde{g}^{\rho m}M^{\nu}-\tilde{g}^{\nu m}M^{\rho}\right)\right) \end{pmatrix} \\ +\frac{1}{2}\begin{pmatrix} \tilde{D}^{\rho}\tilde{D}_m\left(M^{\sigma\mu m}-\frac{1}{2}\left(M^{\sigma m\mu}-M^{\mu m\sigma}\right)+\frac{1}{2}\left(\tilde{g}^{\sigma m}M^{\mu}-\tilde{g}^{\mu m}M^{\sigma}\right)\right) \\ -\tilde{D}^{\sigma}\tilde{D}_m\left(M^{\rho\mu m}-\frac{1}{2}\left(M^{\rho m\mu}-M^{\mu m\rho}\right)+\frac{1}{2}\left(\tilde{g}^{\rho m}M^{\mu}-\tilde{g}^{\mu m}M^{\rho}\right)\right) \end{pmatrix} \\ -2\alpha\frac{\kappa}{G_N}\left(M^{\rho\sigma\mu}-\frac{1}{2}\left(M^{\rho\mu\sigma}-M^{\sigma\mu\rho}\right)+\frac{1}{2}\left(\tilde{g}^{\rho\mu}M^{\sigma}-\tilde{g}^{\sigma\mu}M^{\rho}\right)\right) \end{pmatrix} = \begin{pmatrix} \frac{8\pi\kappa}{c^4}\tilde{s}_e^{\rho\sigma\mu} \\ -\alpha\frac{\kappa}{G_N}\left(\tilde{D}^{\mu}\chi^{\rho\sigma}-\tilde{g}^{\rho\mu}\tilde{D}_m\chi^{\sigma m}+g^{\sigma\mu}\tilde{D}_m\chi^{\rho m}\right) \\ -\alpha\frac{\kappa}{G_N}\left(\tilde{D}^{\mu}\chi^{\rho\sigma}-\left(\tilde{D}^{\rho}\chi^{\mu\sigma}-\tilde{D}^{\sigma}\chi^{\mu\rho}\right)\right) \\ +\alpha\frac{\kappa}{G_N}\begin{pmatrix} \tilde{D}^{\rho}\left(\phi^{\sigma\mu}-\frac{1}{2}\tilde{g}^{\sigma\mu}\phi\right)-\tilde{D}^{\sigma}\left(\phi^{\rho\mu}-\frac{1}{2}\tilde{g}^{\rho\mu}\phi\right) \\ -\tilde{g}^{\rho\mu}\tilde{D}_m\left(\phi^{\sigma m}-\frac{1}{2}\tilde{g}^{\sigma m}\phi\right) \\ +\tilde{g}^{\sigma\mu}\tilde{D}_m\left(\phi^{\rho m}-\frac{1}{2}\tilde{g}^{\to\rho m}\phi\right) \end{pmatrix} \end{pmatrix}$$

......(9.3)

B. When $\kappa_1=1,\kappa_2=-\kappa_3=1,\kappa_4=-\kappa_5=\kappa_6=0,\alpha=0$, the above equation has another simplest form:

$$\begin{pmatrix} \begin{pmatrix} \tilde{D}^{\mu}\tilde{D}_{\nu}\left(M^{\rho\sigma\nu}-\left(M^{\rho\nu\sigma}-M^{\sigma\nu\rho}\right)\right) \\ -\tilde{D}^{\nu}\tilde{D}_{\nu}\left(M^{\rho\sigma\mu}-\left(M^{\rho\mu\sigma}-M^{\sigma\mu\rho}\right)\right) \end{pmatrix} \\ -\tilde{D}_{\nu}\begin{pmatrix} \tilde{g}^{\rho\mu}\tilde{D}_m\left(M^{\sigma\nu m}-\left(M^{\sigma m\nu}-M^{\nu m\sigma}\right)\right) \\ -\tilde{g}^{\sigma\mu}\tilde{D}_m\left(M^{\rho\nu m}-\left(M^{\rho m\nu}-M^{\nu m\rho}\right)\right) \end{pmatrix} \\ +\begin{pmatrix} +\tilde{D}^{\rho}\tilde{D}_m\left(M^{\sigma\mu m}-\left(M^{\sigma m\mu}-M^{\mu m\sigma}\right)\right) \\ -\tilde{D}^{\sigma}\tilde{D}_m\left(M^{\rho\mu m}-\left(M^{\rho m\mu}-M^{\mu m\rho}\right)\right) \end{pmatrix} \\ +2\beta\frac{\kappa}{G_N}\left(\tilde{M}^{\rho\sigma\mu}-\left(\tilde{M}^{\rho\mu\sigma}-\tilde{M}^{\sigma\mu\rho}\right)\right) \end{pmatrix} = \begin{pmatrix} \frac{8\pi\kappa}{c^4}\tilde{s}_e^{\rho\sigma\mu} \\ +2\beta\frac{\kappa}{G_N}\left(\tilde{D}^{\mu}\chi^{\rho\sigma}-\left(\tilde{D}^{\rho}\chi^{\mu\sigma}-\tilde{D}^{\sigma}\chi^{\mu\rho}\right)\right) \end{pmatrix} \text{......(9.4)}$$

(4) Gravitational field equation of motion

$$\begin{pmatrix} \begin{pmatrix} \tilde{D}^{\mu}\tilde{D}_{\rho}\left(\phi^{\nu\rho}-\frac{1}{2}\tilde{g}^{\nu\rho}\phi\right)-\tilde{D}^{\rho}\tilde{D}_{\rho}\left(\phi^{\mu\nu}-\frac{1}{2}\tilde{g}^{\mu\nu}\phi\right) \\ +\tilde{D}^{\nu}\tilde{D}_{\rho}\left(\phi^{\mu\rho}-\frac{1}{2}\tilde{g}^{\mu\rho}\phi\right)-\tilde{g}^{\mu\nu}\tilde{D}_m\tilde{D}_{\rho}\left(\phi^{m\rho}-\frac{1}{2}\tilde{g}^{m\rho}\phi\right) \end{pmatrix} \\ +\beta\left(\tilde{D}^{\mu}\tilde{D}_{\sigma}\chi^{\nu\sigma}-\tilde{D}^{\nu}\tilde{D}_{\sigma}\chi^{\mu\sigma}+\tilde{D}_{\sigma}\tilde{D}^{\sigma}\chi^{\mu\nu}\right) \\ -\beta\tilde{D}_{\sigma}\left(M^{\mu\nu\sigma}-M^{\mu\sigma\nu}+M^{\nu\sigma\mu}\right) \end{pmatrix} = \begin{pmatrix} \frac{8\pi G_N}{c^4}\tilde{P}_e^{\mu\nu} \\ -\alpha\tilde{D}_{\rho}\left(M^{\rho\mu\nu}+\left(\tilde{g}^{\rho\nu}M^{\mu}-\tilde{g}^{\mu\nu}M^{\rho}\right)\right) \end{pmatrix}$$

A. When $\kappa_1 = 1, \kappa_2 = -\kappa_3 = \frac{1}{2}, \kappa_4 = -\kappa_5 = \kappa_6 = -\frac{1}{4}, \beta = -\alpha$,the equation of motion of the gravitational field is:

$$\begin{pmatrix} \begin{pmatrix} \tilde{D}^{\mu}\tilde{D}_{\rho}\left(\phi^{\nu\rho}-\frac{1}{2}\tilde{g}^{\nu\rho}\phi\right)-\tilde{D}^{\rho}\tilde{D}_{\rho}\left(\phi^{\mu\nu}-\frac{1}{2}\tilde{g}^{\mu\nu}\phi\right) \\ +\tilde{D}^{\nu}\tilde{D}_{\rho}\left(\phi^{\mu\rho}-\frac{1}{2}\tilde{g}^{\mu\rho}\phi\right)-\tilde{g}^{\mu\nu}\tilde{D}_{m}\tilde{D}_{\rho}\left(\phi^{m\rho}-\frac{1}{2}\tilde{g}^{m\rho}\phi\right) \end{pmatrix} \\ -\alpha\left(\tilde{D}^{\mu}\tilde{D}_{\sigma}\chi^{\nu\sigma}-\tilde{D}^{\nu}\tilde{D}_{\sigma}\chi^{\mu\sigma}+\tilde{D}_{\sigma}\tilde{D}^{\sigma}\chi^{\mu\nu}\right) \end{pmatrix} = \begin{pmatrix} \frac{8\pi G_N}{c^4}\tilde{P}_e^{\mu\nu} \\ -2\alpha\tilde{D}_{\rho}\begin{pmatrix} M^{\rho\mu\nu}-\frac{1}{2}\left(M^{\rho\nu\mu}-M^{\mu\nu\rho}\right) \\ +\frac{1}{2}\left(\tilde{g}^{\rho\nu}M^{\mu}-\tilde{g}^{\mu\nu}M^{\rho}\right) \end{pmatrix} \end{pmatrix}$$

......(9.5)

B. When $\kappa_1 = 1, \kappa_2 = -\kappa_3 = 1, \kappa_4 = -\kappa_5 = \kappa_6 = 0, \alpha = 0$, the equation of motion of the gravitational field is:

$$\begin{pmatrix} \begin{pmatrix} \tilde{D}^{\mu}\tilde{D}_{\rho}\left(\phi^{\nu\rho}-\frac{1}{2}\tilde{g}^{\nu\rho}\phi\right)-\tilde{D}^{\rho}\tilde{D}_{\rho}\left(\phi^{\mu\nu}-\frac{1}{2}\tilde{g}^{\mu\nu}\phi\right) \\ +\tilde{D}^{\nu}\tilde{D}_{\rho}\left(\phi^{\mu\rho}-\frac{1}{2}\tilde{g}^{\mu\rho}\phi\right)-\tilde{g}^{\mu\nu}\tilde{D}_{m}\tilde{D}_{\rho}\left(\phi^{m\rho}-\frac{1}{2}\tilde{g}^{m\rho}\phi\right) \end{pmatrix} \\ +\beta\left(\tilde{D}^{\mu}\tilde{D}_{\sigma}\chi^{\nu\sigma}-\tilde{D}^{\nu}\tilde{D}_{\sigma}\chi^{\mu\sigma}+\tilde{D}_{\sigma}\tilde{D}^{\sigma}\chi^{\mu\nu}\right) \end{pmatrix} = \begin{pmatrix} \frac{8\pi G_N}{c^4}\tilde{P}_e^{\mu\nu} \\ +\beta\tilde{D}_{\sigma}\left(M^{\mu\nu\sigma}-M^{\mu\sigma\nu}+M^{\nu\sigma\mu}\right) \end{pmatrix}$$ ......(9.6)

Linear approximation of the Gauge Condition

Substituting the linear approximation of the above field physical quantities into the covariant gauge condition in the flat space-time background, we obtain:

（1）Gravitational field gauge condition：

$$\tilde{D}^{\rho}\phi_{\rho\mu} = \frac{1}{2}\tilde{D}_{\mu}\phi, \tilde{D}^{\rho}\chi_{\rho\mu} = 0, \phi = \phi_{\rho}{}^{\rho}$$

......(9.7)

（2）Spin field gauge condition：

A.When $\kappa_1 = 1, \kappa_2 = -\kappa_3 = \frac{1}{2}, \kappa_4 = -\kappa_5 = \kappa_6 = -\frac{1}{4}, \beta = -\alpha$,there is

$$\tilde{D}^{\mu}\left(M_{\rho\sigma\mu}-\frac{1}{2}\left(M_{\rho\mu\sigma}-M_{\sigma\mu\rho}\right)+\frac{1}{2}\left(\tilde{g}_{\rho\mu}M_{\sigma}-\tilde{g}_{\sigma\mu}M_{\rho}\right)\right)=0$$

......(9.8)

B.When $\kappa_1 = 1, \kappa_2 = -\kappa_3 = 1, \kappa_4 = -\kappa_5 = \kappa_6 = 0, \alpha = 0$,there is：

$$\tilde{D}^{\mu}\left(M_{\rho\sigma\mu}-\left(M_{\rho\mu\sigma}-M_{\sigma\mu\rho}\right)\right)=0$$

......(9.9)

Under this gauge condition, the number of gauge conditions is greater than the number of independent equations, and only the unique solution $M_{\rho\sigma\mu} = 0$ mathematically satisfies the equation solution. We think this does not meet the physical requirements and abandon it.

（3）The electromagnetic field gauge condition:

$$\tilde{D}^{\rho}A_{\rho} = 0$$

......(9.10)

Linear Approximate Equation Expressed by Potential Function

From the above discussion of the gauge condition, we obtain that the only undetermined constant satisfying the requirements of the simplest linear approximation is:

$$\kappa_1 = 1, \kappa_2 = -\kappa_3 = \frac{1}{2}, \kappa_4 = -\kappa_5 = \kappa_6 = -\frac{1}{4}, \beta = -\alpha$$

Substituting the above gauge conditions into the equations of motion and omitting $O(\varepsilon)$, we get:

(1) he linear approximation equation of motion of the spin field:

$$\begin{cases} \begin{pmatrix} \tilde{D}^\nu \tilde{D}_\nu \begin{pmatrix} M^{\rho\sigma\mu} - \frac{1}{2}\left(M^{\rho\mu\sigma} - M^{\sigma\mu\rho}\right) \\ +\frac{1}{2}\left(\tilde{g}^{\rho\mu} M^\sigma - \tilde{g}^{\sigma\mu} M^\rho\right) \end{pmatrix} \\ +2\alpha \frac{\kappa}{G_N} \begin{pmatrix} M^{\rho\sigma\mu} - \frac{1}{2}\left(M^{\rho\mu\sigma} - M^{\sigma\mu\rho}\right) \\ +\frac{1}{2}\left(\tilde{g}^{\rho\mu} M^\sigma - \tilde{g}^{\sigma\mu} M^\rho\right) \end{pmatrix} \end{pmatrix} = -\begin{pmatrix} \frac{8\pi\kappa}{c^4} \tilde{s}_e^{\rho\sigma\mu} - \alpha \frac{\kappa}{G_N}\left(2\tilde{D}^\mu \chi^{\rho\sigma} - \left(\tilde{D}^\rho \chi^{\mu\sigma} - \tilde{D}^\sigma \chi^{\mu\rho}\right)\right) \\ +\alpha \frac{\kappa}{G_N}\left(\tilde{D}^\rho\left(\phi^{\sigma\mu} - \frac{1}{2}\tilde{g}^{\sigma\mu}\phi\right) - \tilde{D}^\sigma\left(\phi^{\rho\mu} - \frac{1}{2}\tilde{g}^{\rho\mu}\phi\right)\right) \end{pmatrix} \\ \tilde{D}_\mu\left(M^{\rho\sigma\mu} - \frac{1}{2}\left(M^{\rho\mu\sigma} - M^{\sigma\mu\rho}\right) + \frac{1}{2}\left(\tilde{g}^{\rho\mu} M^\sigma - \tilde{g}^{\sigma\mu} M^\rho\right)\right) = 0 \end{cases}$$

$$\begin{cases} \mathbf{X}^{\rho\sigma\mu} = \left(M^{\rho\sigma\mu} - \frac{1}{2}\left(M^{\rho\mu\sigma} - M^{\sigma\mu\rho}\right) + \frac{1}{2}\left(\tilde{g}^{\rho\mu} M^\sigma - \tilde{g}^{\sigma\mu} M^\rho\right)\right) \\ \mathbf{\Phi}^{\mu\nu} = \left(\phi^{\mu\nu} - \frac{1}{2}\tilde{g}^{\mu\nu}\phi\right) \\ \left(\tilde{D}^\nu \tilde{D}_\nu \mathbf{X}^{\rho\sigma\mu} + 2\alpha \frac{\kappa}{G_N} \mathbf{X}^{\rho\sigma\mu}\right) = -\begin{pmatrix} \frac{8\pi\kappa}{c^4} \tilde{s}_e^{\rho\sigma\mu} + \alpha \frac{\kappa}{G_N}\left(\tilde{D}^\rho \mathbf{\Phi}^{\sigma\mu} - \tilde{D}^\sigma \mathbf{\Phi}^{\rho\mu}\right) \\ -\alpha \frac{\kappa}{G_N}\left(2\tilde{D}^\mu \chi^{\rho\sigma} - \left(\tilde{D}^\rho \chi^{\mu\sigma} - \tilde{D}^\sigma \chi^{\mu\rho}\right)\right) \end{pmatrix} \\ \tilde{D}_\mu \mathbf{X}^{\rho\sigma\mu} = 0 \end{cases}$$

......(9.11)

For $\mathbf{X}^{\rho\sigma\mu} = \left(M^{\rho\sigma\mu} - \frac{1}{2}\left(M^{\rho\mu\sigma} - M^{\sigma\mu\rho}\right) + \frac{1}{2}\left(\tilde{g}^{\rho\mu} M^\sigma - \tilde{g}^{\sigma\mu} M^\rho\right)\right)$ of the above equation to have a physically meaningful real solution at rest, it must satisfy: $\alpha \frac{\kappa}{G_N} \geq 0$。

(2) The linear approximate equation of motion of the gravitational field is:

$$\begin{pmatrix} -\tilde{D}^\rho \tilde{D}_\rho\left(\phi^{\mu\nu} - \frac{1}{2}\tilde{g}^{\mu\nu}\phi\right) \\ -\alpha \tilde{D}_\sigma \tilde{D}^\sigma \chi^{\mu\nu} \end{pmatrix} = \begin{pmatrix} \frac{8\pi G_N}{c^4} \tilde{P}_e^{\mu\nu} \\ -2\alpha \tilde{D}_\rho \begin{pmatrix} M^{\rho\mu\nu} - \frac{1}{2}\left(M^{\rho\nu\mu} - M^{\mu\nu\rho}\right) \\ +\frac{1}{2}\left(\tilde{g}^{\rho\nu} M^\mu - \tilde{g}^{\mu\nu} M^\rho\right) \end{pmatrix} \end{pmatrix}$$

$$\begin{cases} \mathbf{X}^{\rho\mu\nu} = \left(M^{\rho\mu\nu} - \frac{1}{2}\left(M^{\rho\nu\mu} - M^{\mu\nu\rho}\right) + \frac{1}{2}\left(\tilde{g}^{\rho\nu} M^\mu - \tilde{g}^{\mu\nu} M^\rho\right)\right) \\ \mathbf{\Phi}^{\mu\nu} = \left(\phi^{\mu\nu} - \frac{1}{2}\tilde{g}^{\mu\nu}\phi\right) \\ \mathbf{Z}^{\mu\nu} = \tilde{D}_\rho \mathbf{X}^{\mu\rho\nu} + \tilde{D}_\rho \mathbf{X}^{\nu\rho\mu}, \overline{\mathbf{Z}}^{\mu\nu} = \tilde{D}_\rho \mathbf{X}^{\mu\rho\nu} - \tilde{D}_\rho \mathbf{X}^{\nu\rho\mu} \\ \tilde{D}^\rho \tilde{D}_\rho \mathbf{\Phi}^{\mu\nu} = -\left(\frac{4\pi G_N}{c^4}\left(\tilde{P}_e^{\mu\nu} + \tilde{P}_e^{\nu\mu}\right) + \alpha \mathbf{Z}^{\mu\nu}\right) \\ \tilde{D}_\nu \mathbf{\Phi}^{\mu\nu} = 0 \\ \alpha \tilde{D}_\sigma \tilde{D}^\sigma \chi^{\mu\nu} = -\left(\frac{4\pi G_N}{c^4}\left(\tilde{P}_e^{\mu\nu} - \tilde{P}_e^{\nu\mu}\right) + \alpha \overline{\mathbf{Z}}^{\mu\nu}\right) \\ \tilde{D}_\nu \chi^{\mu\nu} = 0 \end{cases}$$

Taking into account $\tilde{s}_e^{\rho\sigma\mu}=-\tilde{s}_e^{\sigma\rho\mu}=-\tilde{s}_e^{\rho\mu\sigma}$, we obtain from the spin-field linear approximation Eq. 9.8:

$$\begin{cases}\left(\tilde{D}^{\nu}\tilde{D}_{\nu}\mathbf{Z}^{\sigma\mu}+2\alpha\dfrac{\kappa}{G_N}\mathbf{Z}^{\sigma\mu}\right)=2\alpha\dfrac{\kappa}{G_N}\tilde{D}^{\rho}\tilde{D}_{\rho}\mathbf{\Phi}^{\sigma\mu}\\ \left(\tilde{D}^{\nu}\tilde{D}_{\nu}\overline{\mathbf{Z}}^{\sigma\mu}+2\alpha\dfrac{\kappa}{G_N}\overline{\mathbf{Z}}^{\sigma\mu}\right)=\left(\dfrac{16\pi\kappa}{c^4}\tilde{D}_{\rho}\tilde{s}_e^{\sigma\mu\rho}-2\alpha\dfrac{\kappa}{G_N}\tilde{D}^{\rho}\tilde{D}_{\rho}\chi^{\sigma\mu}\right)\end{cases}$$

Substituting the equation of motion of the gravitational field into the above equations, and considering $\tilde{D}_{\mu}\tilde{s}^{\rho\sigma\mu}=\tilde{P}_e^{\sigma\rho}-\tilde{P}_e^{\rho\sigma}$, the equation of motion of the gravitational field is obtained as follows:

$$\begin{cases}\left(\tilde{D}^{\nu}\tilde{D}_{\nu}\mathbf{Z}^{\sigma\mu}+2\alpha(1+\alpha)\dfrac{\kappa}{G_N}\mathbf{Z}^{\sigma\mu}\right)=-2\alpha\dfrac{\kappa}{G_N}\dfrac{4\pi G_N}{c^4}\left(\tilde{P}_e^{\mu\nu}+\tilde{P}_e^{\nu\mu}\right)\\ \tilde{D}^{\nu}\tilde{D}_{\nu}\overline{\mathbf{Z}}^{\sigma\mu}=-\dfrac{16\pi\kappa}{c^4}\tilde{D}_{\rho}\tilde{s}_e^{\rho\sigma\mu}\end{cases}$$

Let $\tilde{T}_e^{\mu\nu}=\dfrac{1}{2}\left(\tilde{P}_e^{\mu\nu}+\tilde{P}_e^{\nu\mu}\right),\alpha_L^{\ 2}=2\alpha(1+\alpha)\dfrac{\kappa}{G_N}$, Finally, the equation of motion of the gravitational field is obtained as follows:

$$\begin{cases}\alpha_L^{\ 2}=2\alpha(1+\alpha)\dfrac{\kappa}{G_N},\mathbf{X}^{\rho\mu\nu}=\left(M^{\rho\mu\nu}-\dfrac{1}{2}\left(M^{\rho\nu\mu}-M^{\mu\nu\rho}\right)+\dfrac{1}{2}\left(\tilde{g}^{\rho\nu}M^{\mu}-\tilde{g}^{\mu\nu}M^{\rho}\right)\right)\\ \mathbf{\Phi}^{\mu\nu}=\left(\phi^{\mu\nu}-\dfrac{1}{2}\tilde{g}^{\mu\nu}\phi\right),\mathbf{Z}^{\mu\nu}=\tilde{D}_{\rho}\mathbf{X}^{\mu\rho\nu}+\tilde{D}_{\rho}\mathbf{X}^{\nu\rho\mu},\overline{\mathbf{Z}}^{\mu\nu}=\tilde{D}_{\rho}\mathbf{X}^{\mu\rho\nu}-\tilde{D}_{\rho}\mathbf{X}^{\nu\rho\mu}\\ \tilde{D}_{\rho}\mathbf{\Phi}^{\mu\rho}=0\\ \tilde{D}^{\rho}\tilde{D}_{\rho}\mathbf{Z}^{\mu\nu}+\alpha_L^{\ 2}\mathbf{Z}^{\mu\nu}=-\alpha\dfrac{16\pi\kappa}{c^4}\tilde{T}_e^{\mu\nu}\\ \tilde{D}^{\rho}\tilde{D}_{\rho}\mathbf{\Phi}^{\mu\nu}=-\left(\dfrac{8\pi G_N}{c^4}\tilde{T}_e^{\mu\nu}+\alpha\mathbf{Z}^{\mu\nu}\right)\\ \tilde{D}_{\rho}\chi^{\mu\rho}=0\\ \tilde{D}^{\rho}\tilde{D}_{\rho}\overline{\mathbf{Z}}^{\mu\nu}=-\dfrac{16\pi\kappa}{c^4}\tilde{D}_{\rho}\tilde{s}_e^{\mu\nu\rho}\\ \tilde{D}^{\rho}\tilde{D}_{\rho}\chi^{\mu\nu}=-\left(\dfrac{4\pi G_N}{\alpha c^4}\tilde{D}_{\rho}\tilde{s}^{\mu\nu\rho}+\overline{\mathbf{Z}}^{\mu\nu}\right)\end{cases}$$

......(9.12)

For $Z^{\mu\nu}$ to have a physically meaningful real solution at rest, $\alpha_L^{\ 2}=2\alpha(1+\alpha)\dfrac{\kappa}{G_N}\geq 0$ should be satisfied.In general, the $\tilde{D}_{\rho}\tilde{s}^{\mu\nu\rho}=\tilde{P}_e^{\nu\mu}-\tilde{P}_e^{\mu\nu}\neq 0$ of a 1/2 spin particle,this requires: $\alpha\neq 0$.

（3）Linear approximate equation of motion for electromagnetic field:

$$\begin{cases}\tilde{D}_{\nu}\tilde{D}^{\nu}A^{\mu}=4\pi\tilde{j}_e^{\mu}\\ \tilde{D}_{\rho}A^{\rho}=0\end{cases}$$

......(9.13)

According to the above discussion, for the linear approximation of the gravitational and spin field equations to have physically meaningful real solutions at rest, the undetermined constants must satisfy:

$$\begin{cases} \alpha \dfrac{\kappa}{G_N} \ge 0 \\ \alpha(1+\alpha)\dfrac{\kappa}{G_N} \ge 0 \\ \alpha \ne 0 \end{cases}$$

Therefore, there are two sets of solutions:

（a）The first set of solutions: $\alpha > 0, \kappa > 0$；（b）The second set of solutions: $-1 \le \alpha < 0, \kappa < 0$。

Conclusion

Formulation of ECT Spin-Gravitational Field Theory in Riemannian Background Spacetime

By decomposing the frame field $\lambda_{\mu}^{(\alpha)}, \lambda_{(\alpha)}^{\mu}$ and the frame affine connection $\hat{\Gamma}_{(\beta)\mu}^{(\alpha)}$ of the ECT spin-gravity field theory on a curved spacetime M^4 with torsion into a frame field $\tilde{\lambda}_{\mu}^{(\alpha)}, \tilde{\lambda}_{(\alpha)}^{\mu}$, a frame affine connection $\hat{\tilde{\Gamma}}_{(\beta)\mu}^{(\alpha)}$, a curved deviation tensor $\omega_{\mu}{}^{\nu}, \varpi_{\mu}{}^{\nu}$ and a twisted deviation tensor $M_{\sigma\mu}^{\rho}$ on a Riemannian background spacetime, We can obtain the formulation of ECT spin-gravity field theory in Riemannian background spacetime:

$$\lambda_{\mu}^{(\alpha)} = \omega_{\mu}{}^{\nu}\tilde{\lambda}_{\nu}^{(\alpha)}, \lambda_{(\alpha)}^{\mu} = \tilde{\lambda}_{(\alpha)}^{\nu}\varpi_{\nu}{}^{\mu}, \omega_{\mu}{}^{\rho}\varpi_{\rho}{}^{\nu} = \varpi_{\mu}{}^{\rho}\omega_{\rho}{}^{\nu} = \delta_{\mu}^{\nu}$$
$$\hat{\Gamma}_{(\beta)\mu}^{(\alpha)} = \hat{\tilde{\Gamma}}_{(\beta)\mu}^{(\alpha)} + M_{(\beta)\mu}^{(\alpha)}, M_{(\beta)\mu}^{(\alpha)} = \tilde{\lambda}_{\rho}^{(\alpha)}\tilde{\lambda}_{(\beta)}^{\sigma}M_{\sigma\mu}^{\rho}, A_{\mu} = A_{\mu}$$

……(10.1)

Where A_{μ} is the electromagnetic field potential, $M_{(\beta)\mu}^{(\alpha)}$ is the spin field potential, and $\omega_{\mu}{}^{\nu}, \varpi_{\mu}{}^{\nu}$ is the gravitational field potential.

When the Riemannian background spacetime is flat, we get:

$$\hat{\tilde{\Gamma}}_{(\beta)\mu}^{(\alpha)} = 0, \hat{\tilde{R}}_{(\beta)\mu\nu}^{(\alpha)} = 0, \tilde{R}_{\sigma\mu\nu}^{\rho} = 0$$

……(10.2)

The time-axis separated coordinate system under the flat space-time background is adopted, and the metric of the flat space-time background is:

$$\eta_{(\alpha\beta)} = \begin{pmatrix} 1 & 0 \\ 0 & -\delta_{(ab)} \end{pmatrix}, \eta^{(\alpha\beta)} = \begin{pmatrix} 1 & 0 \\ 0 & -\delta^{(ab)} \end{pmatrix}, \tilde{g}_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & -\tilde{g}_{ij} \end{pmatrix}, \tilde{g}^{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & -\tilde{g}^{ij} \end{pmatrix}$$
$$\tilde{\lambda}_{\mu}^{(\alpha)} = \begin{pmatrix} 1 & 0 \\ 0 & \tilde{\lambda}_{j}^{(a)} \end{pmatrix}, \tilde{\lambda}_{(\alpha)\mu} = \begin{pmatrix} 1 & 0 \\ 0 & -\tilde{\lambda}_{(a)j} \end{pmatrix}, \tilde{\lambda}_{(\alpha)}^{\mu} = \begin{pmatrix} 1 & 0 \\ 0 & \tilde{\lambda}_{(a)}^{j} \end{pmatrix}, \tilde{\lambda}^{(\alpha)\mu} = \begin{pmatrix} 1 & 0 \\ 0 & -\tilde{\lambda}^{(a)j} \end{pmatrix}$$

……(10.3)

The kinetic part of the gravitational field on a flat background spacetime is：

$$\omega_{\mu}{}^{\nu} = \begin{pmatrix} N & N^k\omega_k{}^j \\ \bar{N}_i & \omega_i{}^j \end{pmatrix}, \varpi_{\mu}{}^{\nu} = \begin{pmatrix} \dfrac{1}{N - N^m\bar{N}_m} & -\dfrac{N^j}{N - N^m\bar{N}_m} \\ -\dfrac{\varpi_i{}^k\bar{N}_k}{N - N^m\bar{N}_m} & \varpi_i{}^j + \dfrac{\varpi_i{}^k\bar{N}_k N^j}{N - N^m\bar{N}_m} \end{pmatrix}$$

……(10.4)

The time-axis separated coordinate system is adopted in the generalized ADM background spacetime, and the metric of the generalized ADM background spacetime is:

$$\eta_{(\alpha\beta)}=\begin{pmatrix}1&0\\0&-\delta_{(ab)}\end{pmatrix},\eta^{(\alpha\beta)}=\begin{pmatrix}1&0\\0&-\delta^{(ab)}\end{pmatrix},\tilde{g}_{\mu\nu}=\begin{pmatrix}1&0\\0&-g_{ij}\end{pmatrix},\tilde{g}^{\mu\nu}=\begin{pmatrix}1&0\\0&-g^{ij}\end{pmatrix}$$

$$\tilde{\lambda}_{\mu}^{(\alpha)}=\begin{pmatrix}1&0\\0&\lambda_{j}^{(a)}\end{pmatrix},\tilde{\lambda}_{(\alpha)\mu}=\begin{pmatrix}1&0\\0&-\lambda_{(a)j}\end{pmatrix},\tilde{\lambda}_{(\alpha)}^{\mu}=\begin{pmatrix}1&0\\0&\lambda_{(a)}^{j}\end{pmatrix},\tilde{\lambda}^{(\alpha)\mu}=\begin{pmatrix}1&0\\0&-\lambda^{(a)j}\end{pmatrix}$$

The kinetic part of the generalized ADM background spacetime is:

$$\omega_{\mu}{}^{\nu}=\begin{pmatrix}N&N^{j}\\\bar{N}_{i}&\delta_{i}^{j}\end{pmatrix},\varpi_{\mu}{}^{\nu}=\begin{pmatrix}\frac{1}{N-N^{m}\bar{N}_{m}}&-\frac{N^{j}}{N-N^{m}\bar{N}_{m}}\\-\frac{\bar{N}_{i}}{N-N^{m}\bar{N}_{m}}&\delta_{i}^{j}+\frac{\bar{N}_{i}N^{j}}{N-N^{m}\bar{N}_{m}}\end{pmatrix}$$

......(10.5)

Covariant Gauge Conditions for ECT Spin-Gravitational Field Theory

(1) Covariant Gauge Condition of Gravitational Field:

$$\tilde{D}^{\rho}\omega_{\rho\mu}=\tilde{D}^{\rho}\omega_{\mu\rho}=k\tilde{D}_{\mu}\omega,\omega=\omega_{\rho}{}^{\rho}$$

Where k is chosen such that the equation of motion for the spin field is the simplest.

......(10.3)

（2）Covariant gauge condition for spin field:

$$\tilde{D}^{\rho}\left(M_{(\alpha\beta)\rho}+a\left(M_{(\alpha)\rho(\beta)}-M_{(\beta)\rho(\alpha)}\right)+b\left(\tilde{\lambda}_{(\alpha)\mu}M_{(\beta)}-\tilde{\lambda}_{(\beta)\mu}M_{(\alpha)}\right)\right)=0$$

Where a,b is chosen such that the equations of motion for the spin field are the simplest.

......(10.4)

（3）Covariant gauge condition of electromagnetic field:

$$\tilde{D}^{\rho}A_{\rho}=0$$

......(10.5)

Noncovariant Gauge Conditions for ECT Spin-Gravitational Field Theory

The time-separated coordinate system of the generalized ADM background space-time is actually the inertial coordinate system of the curved space, and the following gauge conditions can be obtained in the time-separated coordinate system.

The Spaces Isotropic Gauge Condition of Gravitational Field:

$$\omega_{\mu}{}^{\nu}=\begin{pmatrix}\Phi&h^{j}\\\bar{h}_{i}&\Omega\delta_{i}^{j}\end{pmatrix},\varpi_{\mu}{}^{\nu}=\begin{pmatrix}\frac{\Omega}{\Phi\Omega-\bar{h}_{k}h^{k}}&-\frac{h^{j}}{\Phi\Omega-\bar{h}_{k}h^{k}}\\-\frac{\bar{h}_{i}}{\Phi\Omega-\bar{h}_{k}h^{k}}&\frac{1}{\Omega}\left(\delta_{i}^{j}+\frac{\bar{h}_{i}h^{j}}{\Phi\Omega-\bar{h}_{k}h^{k}}\right)\end{pmatrix}$$

......(10.6)

The spaces isotropic gauge condition of the gravitational field is applicable to an arbitrary background spacetime. Letting $\bar{h}_{i}=\Omega\left(\chi_{i}+\varphi_{i}\right)=\Omega\bar{\hbar}_{i},h_{i}=\Phi\left(\chi_{i}-\varphi_{i}\right)=\Phi\hbar_{i}$, we can obtain the spaces isotropic gauge condition of the gravitational field denoted by $\left(\Phi,\Omega,\varphi_{i},\chi_{i}\right)$:

$$\omega_\mu{}^\nu = \begin{pmatrix} \Phi & \Phi\hbar^j \\ \Omega\bar{\hbar}_i & \Omega\delta_i^j \end{pmatrix}, \varpi_\mu{}^\nu = \frac{1}{1-\bar{\hbar}_k\hbar^k}\begin{pmatrix} \frac{1}{\Phi} & -\frac{1}{\Omega}\hbar^j \\ -\frac{1}{\Phi}\bar{\hbar}_i & \frac{1}{\Omega}\left(\delta_i^j\left(1-\bar{\hbar}_k\hbar^k\right)+\bar{\hbar}_i\hbar^j\right) \end{pmatrix}$$

......(10.10)

Time-dependent and Radiation Gauge Conditions of Spin Field

（1）Time-dependent gauge condition of spin field

$$M_{(\alpha\beta)0}+a\left(M_{(\alpha)0(\beta)}-M_{(\beta)0(\alpha)}\right)+b\left(\tilde{\lambda}_{(\alpha)0}M_{(\beta)}-\tilde{\lambda}_{(\beta)0}M_{(\alpha)}\right)=0$$

......(10.11)

（2）Radiation gauge condition of spin field

$$\tilde{D}^i\left(M_{(\alpha\beta)i}+a\left(M_{(\alpha)i(\beta)}-M_{(\beta)i(\alpha)}\right)+b\left(\tilde{\lambda}_{(\alpha)i}M_{(\beta)}-\tilde{\lambda}_{(\beta)i}M_{(\alpha)}\right)\right)=0$$

......(10.12)

Time-Dependent and Radiation Gauge Conditions of Electromagnetic Field

（1）Time-dependent gauge condition of electromagnetic field

$$A_0=0$$

......(10.13)

（2）Radiation gauge condition of electromagnetic field

$$\tilde{D}^i A_i=0$$

......(10.14)

Linear Approximation of ECT Spin-Gravitational Field Theory

The linear approximation of ECT spin-gravitational field theory is to obtain the linear approximate equation of motion for $\phi_{\mu\nu}, \chi_{\mu\nu}, M_{\rho\sigma\mu}, A_\mu$ in the flat background space-time.

（1） Dirac electron field equation of motion:

$$\begin{cases} \frac{1}{2}\left(i\hbar\tilde{\gamma}^\mu\tilde{D}_\mu\psi+i\hbar\tilde{D}_\mu(\tilde{\gamma}^\mu\psi)\right)-mc\psi=0 \\ \frac{1}{2}\left(i\hbar\tilde{D}_\mu\bar{\psi}\tilde{\gamma}^\mu+i\hbar\tilde{D}_\mu(\bar{\psi}\tilde{\gamma}^\mu)\right)+mc\bar{\psi}=0 \end{cases}$$

......(10.15)

（2） The linear approximate equation of motion for the gravitational field in its simplest form is:

（$\kappa_1=1,\kappa_2=-\kappa_3=\frac{1}{2},\kappa_4=-\kappa_5=\kappa_6=-\frac{1}{4},\alpha\frac{\kappa}{G_N}>0,\alpha_L{}^2=2\alpha\left(1+\alpha\right)\frac{\kappa}{G_N}$）:

$$\begin{cases} \mathbf{X}^{\rho\mu\nu}=\left(M^{\rho\mu\nu}-\frac{1}{2}\left(M^{\rho\nu\mu}-M^{\mu\nu\rho}\right)+\frac{1}{2}\left(\tilde{g}^{\rho\nu}M^\mu-\tilde{g}^{\mu\nu}M^\rho\right)\right) \\ \mathbf{\Phi}^{\mu\nu}=\left(\phi^{\mu\nu}-\frac{1}{2}\tilde{g}^{\mu\nu}\phi\right),\mathbf{Z}^{\mu\nu}=\tilde{D}_\rho\mathbf{X}^{\mu\rho\nu}+\tilde{D}_\rho\mathbf{X}^{\nu\rho\mu},\bar{\mathbf{Z}}^{\mu\nu}=\tilde{D}_\rho\mathbf{X}^{\mu\rho\nu}-\tilde{D}_\rho\mathbf{X}^{\nu\rho\mu} \\ \tilde{D}_\rho\mathbf{\Phi}^{\mu\rho}=0 \\ \tilde{D}^\rho\tilde{D}_\rho\mathbf{Z}^{\mu\nu}+\alpha_L{}^2\mathbf{Z}^{\mu\nu}=-\alpha\frac{16\pi\kappa}{c^4}\tilde{T}_e^{\mu\nu} \end{cases}$$

$$\begin{cases}\tilde{D}^{\rho}\tilde{D}_{\rho}\mathbf{\Phi}^{\mu\nu}=-\left(\frac{8\pi G_N}{c^4}\tilde{T}_e^{\mu\nu}+\alpha\mathbf{Z}^{\mu\nu}\right)\\ \tilde{D}_{\rho}\chi^{\mu\rho}=0\\ \tilde{D}^{\rho}\tilde{D}_{\rho}\overline{\mathbf{Z}}^{\mu\nu}=-\frac{16\pi\kappa}{c^4}\tilde{D}_{\rho}\tilde{s}_e^{\mu\nu\rho}\\ \tilde{D}^{\rho}\tilde{D}_{\rho}\chi^{\mu\nu}=-\left(\frac{4\pi G_N}{\alpha c^4}\tilde{D}_{\rho}\tilde{s}^{\mu\nu\rho}+\overline{\mathbf{Z}}^{\mu\nu}\right)\end{cases}$$

......(10.16)

For $Z^{\mu\nu}$ to have a physically meaningful real solution at rest, $\alpha_L{}^2=2\alpha(1+\alpha)\frac{\kappa}{G_N}\geq 0$ should be satisfied. In general,

$\tilde{D}_{\rho}\tilde{s}^{\mu\nu\rho}=\tilde{P}_e^{\nu\mu}-\tilde{P}_e^{\mu\nu}\neq 0$ for 1/2 spin particles, which requires

$\alpha\neq 0$.

（3）The linear approximate equation of motion for the spin field in its simplest form

（$\kappa_1=1,\kappa_2=-\kappa_3=\frac{1}{2},\kappa_4=-\kappa_5=\kappa_6=-\frac{1}{4},\beta=-\alpha$）:

$$\begin{cases}\mathbf{X}^{\rho\sigma\mu}=\left(M^{\rho\sigma\mu}-\frac{1}{2}\left(M^{\rho\mu\sigma}-M^{\sigma\mu\rho}\right)+\frac{1}{2}\left(\tilde{g}^{\rho\mu}M^{\sigma}-\tilde{g}^{\sigma\mu}M^{\rho}\right)\right)\\ \mathbf{\Phi}^{\mu\nu}=\left(\phi^{\mu\nu}-\frac{1}{2}\tilde{g}^{\mu\nu}\phi\right)\\ \left(\tilde{D}^{\nu}\tilde{D}_{\nu}\mathbf{X}^{\rho\sigma\mu}+2\alpha\frac{\kappa}{G_N}\mathbf{X}^{\rho\sigma\mu}\right)=-\begin{pmatrix}\frac{8\pi\kappa}{c^4}\tilde{s}_e^{\rho\sigma\mu}+\alpha\frac{\kappa}{G_N}\left(\tilde{D}^{\rho}\mathbf{\Phi}^{\sigma\mu}-\tilde{D}^{\sigma}\mathbf{\Phi}^{\rho\mu}\right)\\ -\alpha\frac{\kappa}{G_N}\left(2\tilde{D}^{\mu}\chi^{\rho\sigma}-\left(\tilde{D}^{\rho}\chi^{\mu\sigma}-\tilde{D}^{\sigma}\chi^{\mu\rho}\right)\right)\end{pmatrix}\\ \tilde{D}_{\mu}\mathbf{X}^{\rho\sigma\mu}=0\end{cases}$$

......(10.17)

For $\mathbf{X}^{\rho\sigma\mu}$ to have a physically meaningful real solution at rest, $2\alpha\frac{\kappa}{G_N}\geq 0$ should be satisfied

（4）The linear approximation equation of motion for the electromagnetic field:

$$\begin{cases}\tilde{D}_{\nu}\tilde{D}^{\nu}A^{\mu}=4\pi\tilde{j}_e^{\mu}\\ \tilde{D}^{\rho}A_{\rho}=0\end{cases}$$

......(10.18)

Finally, we obtain the undetermined constant as：

$$\begin{pmatrix}\beta_1=-\beta_2=1,\beta_3=0,\alpha''=0,\gamma=0,\beta=-\alpha\\ \kappa_1=1,\kappa_2=-\kappa_3=\frac{1}{2},\kappa_4=-\kappa_5=\kappa_6=-\frac{1}{4}\\ (\alpha>0,\kappa>0)or(-1\leq\alpha<0,\kappa<0)\end{pmatrix}$$

$$\begin{pmatrix}\omega_{\mu\nu}=\tilde{g}_{\mu\nu}+h_{\mu\nu},h_{\mu\nu}=\phi_{\mu\nu}+\chi_{\mu\nu}\\ \phi_{\mu\nu}=\phi_{\nu\mu},\chi_{\mu\nu}=-\chi_{\nu\mu},M_{\rho\sigma\mu}=-M_{\sigma\rho\mu}\\ \phi=\tilde{g}^{\mu\nu}\phi_{\mu\nu},M_{\mu}=\tilde{g}^{\rho\sigma}M_{\mu\rho\sigma}\end{pmatrix}$$

The linear approximation of the ECT spin-gravitation field theory is applicable to the weak field case.

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