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From Ramanujan to Riemann Hypothesis

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Abstract

In this work we offer a careful framework for approaching the critical-line problem associated with the Riemann zeta function. At its heart is a long-standing divide in the subject. On one side are analytic approaches, which study the completed zeta function through its reflection symmetry. On the other side are arithmetic approaches, where related criteria often appear through extremal behavior in divisor functions. The purpose of this paper is not to claim a proof of the Riemann Hypothesis, but to place these two perspectives into a clearer and more usable relationship. The argument begins with reflected analytic data for the completed zeta function. It shows that such data can be described through an odd analytic perturbation, giving a more organized way to understand the analytic side of the problem. This also resolves a common point of confusion: the full complex defect is not required to vanish on the critical line. What matters is more subtle. Under a natural real-symmetry condition, the real part of the defect vanishes on the critical line, and this is the feature that becomes useful for the bridge argument. The arithmetic side is built around Ramanujan's logarithmic divisor profile. The paper establishes the existence and positivity of the relevant extremal scale in the range needed for the proposed connection. These analytic and arithmetic pieces are then brought together through a real bridge functional consisting of a main sign term and a correction term. The main outcome is a conditional criterion for the critical line. If the bridge functional is zero-adapted at the nontrivial zeros, if the real analytic defect satisfies the required one-sided sign condition, and if the correction term remains strictly smaller than the main term, then every nontrivial zero must lie on the critical line. The contribution of this work is therefore structural rather than conclusive. It does not present the Riemann Hypothesis as solved. Instead, it separates what is already established from what still needs to be proved. The key sign law, the domination estimate, and the zero-adaptation identity remain open requirements for any future application of the framework. Its practical value is that it gives researchers a precise checklist for testing whether a proposed analytic or arithmetic strategy can genuinely support a critical-line argument.

Keywords: Riemann Hypothesis, Critical Line, Reflection Symmetry, Analytic Defect, Ramanujan Divisor Profile

Introduction

The Riemann Hypothesis grew out of Riemann's study of the prime-counting problem and remains one of the central open questions in mathematics [1]. In broad terms, it predicts that the nontrivial zeros of the zeta function are arranged on a single vertical line, and this predicted geometry is deeply connected with the error term in prime number distribution. Standard analytic treatments develop the completed zeta function, its functional equation, and the analytic continuation needed to study the critical strip [2]. Edwards gives a historically oriented account of the same circle of ideas, while Ivić records many of the analytic estimates surrounding the zeta function [3,4]. Large-scale computation has verified the predicted line placement for an enormous initial range of zeros, but such verification is not a proof for all zeros [5].

A second line of research reformulates the problem through arithmetic inequalities. Lagarias gave an elementary criterion equivalent to the Riemann Hypothesis [6]. Nicolas and Robin produced criteria involving extremal behavior of arithmetic functions closely related to divisor sums [7,8]. These results show that the critical-line problem is not only a question about complex analysis; it can also be expressed as a problem about sharp arithmetic growth. Ramanujan's work is

naturally relevant to this interface. His formulas for the completed zeta-side functions belong to the analytic side [9], while his theory of highly composite numbers introduced a systematic language for divisor extremality [10]. Later work clarified the structure of highly composite and related numbers in greater detail.

The present paper develops a conditional bridge between these two viewpoints. The point is not to offer an unconditional proof of the Riemann Hypothesis. Rather, the purpose is to isolate exactly what must be proved for a Ramanujan-inspired analytic-arithmetic mechanism to force the critical line. This distinction is important because many proposed approaches to the problem confuse reflection symmetry with coercivity. A symmetric expression can vanish or change sign on a line, but symmetry alone does not imply that the expression has a fixed sign on either side. The framework below keeps these issues separate.

The analytic datum is a function T satisfying

$$T(s) + T(1 - s) = \xi(s). \quad (1.1)$$

The associated defect is

$$D_T(s) = T(s) - T(1 - s). \quad (1.2)$$

It follows that $D_T(1-s) = -D_T(s)$. Since $1-(1/2+it)=1/2-it$, this identity relates two reflected points on the critical line; it does not imply that $D_T(1/2 + it)$ itself is zero. Under real-admissibility, the correct vanishing statement is

$$\operatorname{Re} D_T\left(\frac{1}{2} + it\right) = 0. \quad (1.3)$$

For that reason the bridge in this paper is built from the real defect $\operatorname{Re} D_T$, not from the full complex defect.

The arithmetic datum is Ramanujan's logarithmic divisor profile

$$\Lambda_\varepsilon(N) = \log d(N) - \varepsilon \log N, \quad (1.4)$$

where $d(N)$ is the divisor function. For $0 < \varepsilon < 1$, this profile has a positive global maximum. The largest maximizer is a superior highly composite integer, and the maximum value gives a positive arithmetic weight L_ε . The bridge functional is then defined by

$$\mathcal{B}_\varepsilon(\sigma, t) = L_\varepsilon \operatorname{Re} D_T(\sigma + it) + R_\varepsilon(\sigma, t). \quad (1.5)$$

The proof mechanism is transparent. If this bridge vanishes at every nontrivial zero, while its sign is forced to agree with $\sigma - 1/2$ away from the critical line, then a zero off the critical line is impossible. The principal conclusion is therefore a conditional criterion whose hypotheses are explicit: a one-sided sign law for the real defect, a strict domination estimate for the correction, and a zero-adapted identity at the zeros. The framework separates these three burdens from the identities that can already be proved.

A quantitative map of the notation used in the paper is shown in **Table 1**. It contains ten objects: five analytic-side objects, three arithmetic objects, and two bridge-level objects. The analytic entries define the reflected datum, its real defect, and its normalized form; the arithmetic entries identify the divisor profile and its positive extremal weight; the bridge entries record the correction and the final expression whose sign is tested. The table fixes the notation before the proofs begin and distinguishes structural, extremal, and conditional objects.

Symbol	Definition	Role
$\xi(s)$	Completed zeta function $\frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s)$	Carries the functional equation $\xi(s) = \xi(1-s)$.
$T(s)$	Reflected datum satisfying $T(s) + T(1-s) = \xi(s)$	Splits the completed function into two reflected pieces.
$D_T(s)$	$T(s) - T(1-s)$	Measures the reflected imbalance of the analytic datum.
$\text{Re}D_T(\sigma + it)$	Real part of the defect	Vanishes on the critical line under real-admissibility.
$A_T(\sigma, t)$	$\frac{\text{Re}D_T(\sigma + it)}{\sigma - 1/2}$ for $\sigma \neq 1/2$	Normalized real defect; its positivity is equivalent to the analytic sign law.
$d(N)$	Number of positive divisors of N	Arithmetic input from Ramanujan's theory of highly composite numbers.
$\Lambda_\varepsilon(N)$	$\log d(N) - \varepsilon \log N$	Logarithmic divisor profile; its maximum selects an extremal scale.
L_ε	$\max_{N \in \mathbb{N}} \Lambda_\varepsilon(N)$ for $0 < \varepsilon < 1$	Positive arithmetic weight in the bridge.
$R_\varepsilon(\sigma, t)$	Real correction term	The part that must be dominated by the main term.
$\mathcal{B}_\varepsilon(\sigma, t)$	$L_\varepsilon \text{Re}D_T(\sigma + it) + R_\varepsilon(\sigma, t)$	Bridge whose zero-adaptation and sign barrier imply the critical-line conclusion.

Table 1: Principal Objects and Their Roles in the Bridge Criterion.

The Completed Zeta Function and Reflected Data

The analytic side begins with the completed zeta function

$$\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s). \quad (2.1)$$

This normalization removes the pole of $\zeta(s)$ at $s=1$ and places the functional equation in its symmetric form. The function ξ is entire, satisfies

$$\xi(s) = \xi(1-s), \quad (2.2)$$

and has the real symmetry

$$\xi(\bar{s}) = \overline{\xi(s)}. \quad (2.3)$$

The zeros of ξ are precisely the nontrivial zeros of ζ , counted with multiplicity.

Definition 1 (Admissible reflected datum). Let $U \subset \mathbb{C}$ be a connected open set containing the critical strip $0 < \text{Re}(s) < 1$. Assume that U is invariant under $s \mapsto 1-s$ and $s \mapsto \bar{s}$. An analytic function $T: U \rightarrow \mathbb{C}$ is an admissible reflected datum if

$$T(s) + T(1-s) = \xi(s) \quad (s \in U). \quad (2.4)$$

Definition 2 (Real-admissibility). An admissible reflected datum is real-admissible if

$$T(\bar{s}) = \overline{T(s)} \quad (s \in U). \quad (2.5)$$

Definition 3 (Reflected analytic defect). The defect attached to T is

$$D_T(s) = T(s) - T(1 - s). \quad (2.6)$$

Proposition 4 (Classification of reflected data). The admissible reflected data on U are exactly the functions

$$T(s) = \frac{1}{2}\xi(s) + H(s), \quad (2.7)$$

where H is analytic on U and satisfies

$$H(1 - s) = -H(s). \quad (2.8)$$

For such a datum,

$$D_T(s) = 2H(s). \quad (2.9)$$

Moreover, T is real-admissible if and only if

$$H(\bar{s}) = \overline{H(s)}. \quad (2.10)$$

Proof. Suppose first that T is admissible and put

$$H(s) = T(s) - \frac{1}{2}\xi(s). \quad (2.11)$$

Using (2.2) and (2.4), one obtains

$$H(1 - s) = T(1 - s) - \frac{1}{2}\xi(1 - s) = \xi(s) - T(s) - \frac{1}{2}\xi(s) = -\left(T(s) - \frac{1}{2}\xi(s)\right) = -H(s). \quad (2.12)$$

Thus T has the form (2.7) with H satisfying (2.8). Conversely, if T is defined by (2.7) and H satisfies (2.8), then

$$T(s) + T(1 - s) = \frac{1}{2}\xi(s) + H(s) + \frac{1}{2}\xi(1 - s) + H(1 - s) = \xi(s), \quad (2.13)$$

so T is admissible. Subtracting $T(1 - s)$ from $T(s)$ gives (2.9). The final assertion follows from (2.3) and the identity $H = T - \xi/2$.

Proposition 5 (Recovery from the defect). For every admissible datum T ,

$$T(s) = \frac{1}{2}(\xi(s) + D_T(s)), \quad T(1 - s) = \frac{1}{2}(\xi(s) - D_T(s)). \quad (2.14)$$

Thus the pair (ξ, D_T) determines T .

Proof. Add and subtract the identities

$$\xi(s) = T(s) + T(1 - s), \quad D_T(s) = T(s) - T(1 - s). \quad (2.15)$$

Proposition 6 (Symmetry of the defect). For every admissible datum T ,

$$D_T(1-s) = -D_T(s). \quad (2.16)$$

If T is real-admissible, then

$$D_T(\bar{s}) = \overline{D_T(s)}. \quad (2.17)$$

Consequently, for $s = \sigma + it$,

$$\operatorname{Re} D_T(1 - \sigma + it) = -\operatorname{Re} D_T(\sigma + it), \quad (2.18)$$

and hence

$$\operatorname{Re} D_T\left(\frac{1}{2} + it\right) = 0 \quad (t \in \mathbb{R}). \quad (2.19)$$

Proof. Equation (2.16) follows immediately from (2.6):

$$D_T(1-s) = T(1-s) - T(s) = -D_T(s). \quad (2.20)$$

If T is real-admissible, then

$$D_T(\bar{s}) = T(\bar{s}) - T(1-\bar{s}) = \overline{T(s)} - \overline{T(1-s)} = \overline{D_T(s)}. \quad (2.21)$$

Let $s = \sigma + it$. Since $1 - \bar{s} = 1 - \sigma + it$, equations (2.16) and (2.17) give

$$D_T(1 - \sigma + it) = D_T(1 - \bar{s}) = -D_T(\bar{s}) = -\overline{D_T(s)}. \quad (2.22)$$

Taking real parts proves (2.18). Setting $\sigma = 1/2$ proves (2.19).

Remark 7. The full defect generally does not vanish on the critical line. Reflected symmetry and real-admissibility imply exactly (2.19). This distinction is essential for the bridge constructed below.

Example 8. Let

$$H(s) = s - \frac{1}{2}, \quad T(s) = \frac{1}{2}\xi(s) + s - \frac{1}{2}. \quad (2.23)$$

Then $H(1-s) = -H(s)$, so T is admissible and real-admissible. The defect is

$$D_T(s) = 2s - 1. \quad (2.24)$$

On the critical line,

$$D_T\left(\frac{1}{2} + it\right) = 2it, \quad (2.25)$$

which is nonzero unless $t = 0$. However,

$$\operatorname{Re} D_T(\sigma + it) = 2\sigma - 1, \quad (2.26)$$

so the real defect vanishes exactly on $\sigma = 1/2$ and has the sign of $\sigma - 1/2$.

Ramanujan's Divisor Profile

Let

$$d(N) = \sum_{m|N} 1 \quad (3.1)$$

be the divisor function. For $\varepsilon > 0$, define

$$F_\varepsilon(N) = \frac{d(N)}{N^\varepsilon}, \quad \Lambda_\varepsilon(N) = \log F_\varepsilon(N) = \log d(N) - \varepsilon \log N. \quad (3.2)$$

This is the logarithmic profile associated with Ramanujan's superior highly composite condition. The extremal behavior of such divisor functions is part of the classical tradition developed further by Alaoglu and Erdős and by Gronwall [11,12].

Definition 9 (Superior highly composite number for a parameter). A positive integer N is superior highly composite for the parameter $\varepsilon > 0$ if

$$F_\varepsilon(M) \leq F_\varepsilon(N) \quad (M \leq N), \quad (3.3)$$

and

$$F_\varepsilon(M) < F_\varepsilon(N) \quad (M > N). \quad (3.4)$$

Equivalently, the same inequalities hold with F_ε replaced by Λ_ε .

Lemma 10 (Logarithmic form of the superior condition). A positive integer N is superior highly composite for the parameter $\varepsilon > 0$ if and only if

$$\Lambda_\varepsilon(M) \leq \Lambda_\varepsilon(N) \quad (M \leq N), \quad \Lambda_\varepsilon(M) < \Lambda_\varepsilon(N) \quad (M > N). \quad (3.5)$$

Proof. The logarithm is strictly increasing on $(0, \infty)$. Taking logarithms in

$$\frac{d(M)}{M^\varepsilon} \leq \frac{d(N)}{N^\varepsilon} \quad (3.6)$$

gives

$$\log d(M) - \varepsilon \log M \leq \log d(N) - \varepsilon \log N. \quad (3.7)$$

The strict inequality for $M > N$ is treated in the same way. Exponentiating proves the converse.

The elementary estimate used below is standard in analytic number theory [13].

Lemma 11 (Elementary divisor bound). For every $\delta > 0$ there is a constant $C_\delta > 0$ such that

$$d(N) \leq C_\delta N^\delta \quad (N \geq 1). \quad (3.8)$$

Proof. Write $N = \prod_p p^{a_p}$, where only finitely many exponents are nonzero. Choose P so that $p^\delta \geq 2$ for every prime $p \geq P$. For $p \geq P$ and $a \geq 0$,

$$a + 1 \leq 2^a \leq p^{a\delta}. \quad (3.9)$$

For each prime $p < P$, the number

$$C_{p,\delta} = \sup_{a \geq 0} (a + 1)p^{-a\delta} \quad (3.10)$$

is finite because the exponential factor $p^{-a\delta}$ dominates the linear factor $a + 1$. Therefore

$$a_p + 1 \leq C_{p,\delta} p^{a_p \delta} \quad (p < P), \quad a_p + 1 \leq p^{a_p \delta} \quad (p \geq P). \quad (3.11)$$

Multiplying over the prime divisors of N gives

$$d(N) = \prod_{p|N} (a_p + 1) \leq \left(\prod_{p < P} C_{p,\delta} \right) N^\delta =: C_\delta N^\delta. \quad (3.12)$$

Proposition 12 (Existence and positivity of the arithmetic scale). For every $\varepsilon > 0$, the function F_ε attains a global maximum on $\mathbb{N}$. Let N_ε be the largest integer at which this maximum is attained. Then N_ε is superior highly composite for the parameter ε .

If $0 < \varepsilon < 1$, then

$$L_\varepsilon := \Lambda_\varepsilon(N_\varepsilon) > 0. \quad (3.13)$$

Proof. Apply Lemma 11 with $\delta = \varepsilon/2$. Then

$$F_\varepsilon(N) = \frac{d(N)}{N^\varepsilon} \leq C_{\varepsilon/2} N^{-\varepsilon/2} \rightarrow 0 \quad (N \rightarrow \infty). \quad (3.14)$$

Since $F_\varepsilon(1)=1$, there is an integer K such that $F_\varepsilon(N) < 1$ for all $N > K$. Hence every global maximizer lies in the finite set $\{1, \dots, K\}$, so a global maximum exists. Choose N_ε to be the largest maximizer. Maximality gives (3.3). If equality held in (3.4) for some $M > N_\varepsilon$, then M would be a larger maximizer, a contradiction. Thus N_ε is superior highly composite for ε .

If $0 < \varepsilon < 1$, then

$$F_\varepsilon(2) = 2^{1-\varepsilon} > 1 = F_\varepsilon(1). \quad (3.15)$$

The global maximum is therefore greater than 1, and $\Lambda_\varepsilon(N_\varepsilon) = \log F_\varepsilon(N_\varepsilon) > 0$.

Corollary 13. For $0 < \varepsilon < 1$, N_ε is highly composite:

$$d(M) < d(N_\varepsilon) \quad (1 \leq M < N_\varepsilon). \quad (3.16)$$

Proof. If $M < N_\varepsilon$, then (3.3) gives

$$\frac{d(M)}{M^\varepsilon} \leq \frac{d(N_\varepsilon)}{N_\varepsilon^\varepsilon}. \quad (3.17)$$

Thus

$$d(M) \leq d(N_\varepsilon) \left(\frac{M}{N_\varepsilon} \right)^\varepsilon < d(N_\varepsilon). \quad (3.18)$$

Remark 14. The existence of L_ε uses only the elementary estimate $d(N) = O_\delta(N^\delta)$. No prime number theorem or zero-free region is required at this stage.

Figure 1 shows a quantitative illustration of the divisor profile in the finite range $2 \leq N \leq 400$ for the parameter $\varepsilon = 1/4$. The plot contains 399 evaluated values of $\Lambda_{1/4}(N)$, but only 12 of them form new running records. These record values occur at

$$2, 4, 6, 12, 24, 36, 48, 60, 120, 180, 240, 360. \quad (3.19)$$

The record level rises from approximately 0.5199 at $N = 2$ to approximately 1.7065 at $N = 360$. Fewer than four percent of the plotted integers set a new record in this range. This sparsity explains why the bridge uses an extremal arithmetic scale rather than the full divisor sequence. The figure is explanatory; the proof itself uses the exact global maximizer N_ε .

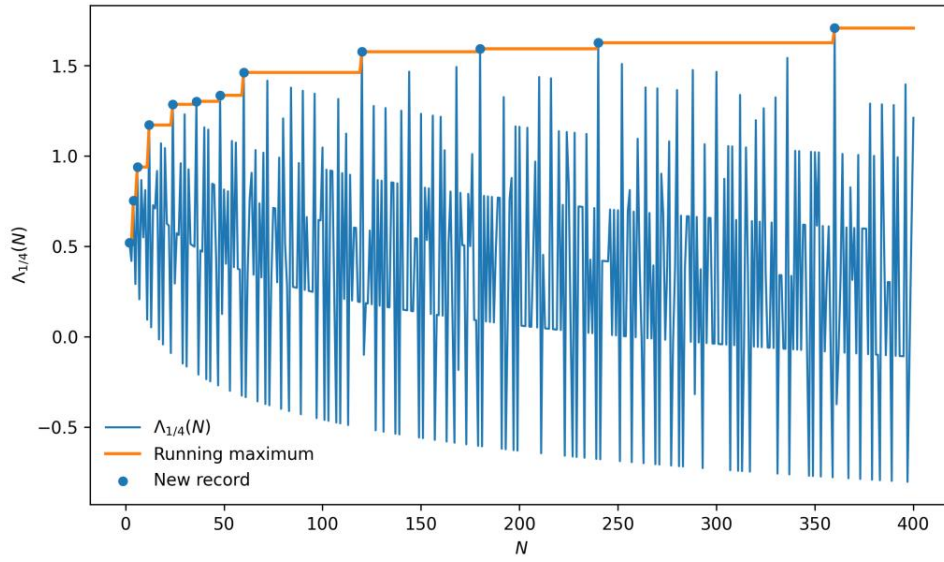


Figure 1: The profile $\Lambda_{1/4}(N) = \log d(N) - \frac{1}{4} \log N$ for $2 \leq N \leq 400$.

The Real Bridge Functional

The analytic construction supplies the real, reflection-odd quantity $\text{Re}D_T$. The arithmetic construction supplies the positive scalar L_ε . A bridge functional is obtained by adding a correction term whose size and zero behavior are to be controlled.

Definition 15 (Real bridge). Let T be a real-admissible reflected datum, let $0 < \varepsilon < 1$, and let L_ε be defined by (3.13). A real bridge attached to (T, ε) is a function

$$\mathcal{B}_\varepsilon: (0,1) \times \mathbb{R} \rightarrow \mathbb{R} \quad (4.1)$$

of the form

$$\mathcal{B}_\varepsilon(\sigma, t) = L_\varepsilon \text{Re}D_T(\sigma + it) + R_\varepsilon(\sigma, t), \quad (4.2)$$

where

$$R_\varepsilon: (0,1) \times \mathbb{R} \rightarrow \mathbb{R}$$

is a correction term.

Remark 16 (Main term and correction). The term $L_\varepsilon \text{Re}D_T(\sigma + it)$ is the part whose sign is meant to be controlled. The correction R_ε is kept separate because it must be estimated rather than absorbed into the datum T . Without a quantitative bound on R_ε , an arbitrary correction could make the bridge vanish or change sign away from the critical line.

For $0 < \sigma < 1$, $\sigma \neq 1/2$, and $t \in \mathbb{R}$, define the normalized real defect by

$$A_T(\sigma, t) := \frac{\text{Re}D_T(\sigma + it)}{\sigma - 1/2}.$$

Thus the domain of A_T consists of all pairs (σ, t) with $0 < \sigma < 1$, $\sigma \neq 1/2$, and $t \in \mathbb{R}$.

Proposition 17 (Normalized form of the bridge). Off the critical line, the bridge can be written as

$$\mathcal{B}_\varepsilon(\sigma, t) = \left(\sigma - \frac{1}{2}\right) \left(L_\varepsilon A_T(\sigma, t) + \frac{R_\varepsilon(\sigma, t)}{\sigma - 1/2}\right), \quad \sigma \neq \frac{1}{2}. \quad (4.3)$$

Proof. By the definition of A_T ,

$$\text{Re}D_T(\sigma + it) = \left(\sigma - \frac{1}{2}\right) A_T(\sigma, t) \quad (\sigma \neq 1/2). \quad (4.4)$$

Substitution in (4.2) gives (4.3).

Definition 18 (Critical normalization). The bridge is critically normalized if

$$R_\varepsilon\left(\frac{1}{2}, t\right) = 0 \quad (t \in \mathbb{R}). \quad (4.5)$$

Proposition 19 (Critical-line vanishing). If the bridge is critically normalized, then

$$\mathcal{B}_\varepsilon\left(\frac{1}{2}, t\right) = 0 \quad (t \in \mathbb{R}). \quad (4.6)$$

Proof. By (2.19),

$$\operatorname{Re} D_T\left(\frac{1}{2} + it\right) = 0. \quad (4.7)$$

Combining this identity with (4.5) and substituting in (4.2) gives (4.6).

Definition 20 (Reflection-odd correction). The correction term is reflection-odd if

$$R_\varepsilon(1 - \sigma, t) = -R_\varepsilon(\sigma, t) \quad (0 < \sigma < 1, t \in \mathbb{R}). \quad (4.8)$$

Proposition 21 (Oddness of the bridge). If R_ε is reflection-odd, then it is critically normalized and

$$\mathcal{B}_\varepsilon(1 - \sigma, t) = -\mathcal{B}_\varepsilon(\sigma, t) \quad (0 < \sigma < 1, t \in \mathbb{R}). \quad (4.9)$$

Proof. Setting $\sigma = 1/2$ in (4.8) gives $R_\varepsilon(1/2, t) = -R_\varepsilon(1/2, t)$, hence $R_\varepsilon(1/2, t) = 0$. Thus the bridge is critically normalized.

For general σ , equations (2.18), (4.2), and (4.8) give

$$\mathcal{B}_\varepsilon(1 - \sigma, t) = L_\varepsilon \operatorname{Re} D_T(1 - \sigma + it) + R_\varepsilon(1 - \sigma, t) = -L_\varepsilon \operatorname{Re} D_T(\sigma + it) - R_\varepsilon(\sigma, t) = -\mathcal{B}_\varepsilon(\sigma, t). \quad (4.10)$$

Remark 22. Reflection-oddness is a structural property. The critical-line criterion below does not require it separately; the criterion uses the stronger analytic sign condition and the domination estimate introduced next.

The Sign Barrier

Reflected symmetry alone gives no sign. A critical-line criterion needs a barrier that excludes zeros of the bridge away from $\sigma = 1/2$.

Definition 23 (Analytic sign condition). A real-admissible datum T satisfies the analytic sign condition on the critical strip if

$$\operatorname{sgn}(\operatorname{Re} D_T(\sigma + it)) = \operatorname{sgn}\left(\sigma - \frac{1}{2}\right) \quad (5.1)$$

for every $0 < \sigma < 1$, $\sigma \neq 1/2$, and $t \in \mathbb{R}$.

Definition 24 (Dominated correction). For $0 < \theta < 1$, the correction term is θ -dominated if

$$|R_\varepsilon(\sigma, t)| \leq \theta L_\varepsilon |\operatorname{Re} D_T(\sigma + it)| \quad (5.2)$$

for every $0 < \sigma < 1$, $\sigma \neq 1/2$, and $t \in \mathbb{R}$.

Remark 25 (Coercive meaning of domination). The restriction $\theta < 1$ provides the strict margin in the argument. With $\theta = 1$, the correction could cancel the main term exactly; with $\theta > 1$, it could reverse the sign. Thus (5.2) is the coercive estimate on which the criterion rests.

Proposition 26 (Quantitative non-vanishing estimate). Assume that R_ε satisfies (5.2) for some $0 < \theta < 1$. Then, for every $0 < \sigma < 1$, $\sigma \neq 1/2$, and $t \in \mathbb{R}$,

$$|\mathcal{B}_\varepsilon(\sigma, t)| \geq (1 - \theta)L_\varepsilon |\operatorname{Re} D_T(\sigma + it)|. \quad (5.3)$$

In particular, wherever $\operatorname{Re} D_T(\sigma + it) \neq 0$, the bridge cannot vanish.

Proof. Put $M = L_\varepsilon \operatorname{Re} D_T(\sigma + it)$. Since $0 < \varepsilon < 1$, Proposition 12 gives $L_\varepsilon > 0$. Equation (4.2) gives $B_\varepsilon = M + R_\varepsilon$, while (5.2) gives $|R_\varepsilon| \leq \theta|M|$. The reverse triangle inequality therefore yields

$$|M + R_\varepsilon| \geq |M| - |R_\varepsilon| \geq (1 - \theta)|M|. \quad (5.4)$$

This is (5.3). If $\operatorname{Re} D_T(\sigma + it) \neq 0$, then $(1 - \theta)L_\varepsilon |\operatorname{Re} D_T(\sigma + it)| > 0$, so the bridge is nonzero.

Proposition 27 (Sign inheritance). Assume that T satisfies (5.1) and that R_ε satisfies (5.2) for some $0 < \theta < 1$. Then

$$\operatorname{sgn}(\mathcal{B}_\varepsilon(\sigma, t)) = \operatorname{sgn}\left(\sigma - \frac{1}{2}\right) \quad (5.5)$$

for every $0 < \sigma < 1$, $\sigma \neq 1/2$, and $t \in \mathbb{R}$. In particular,

$$\mathcal{B}_\varepsilon(\sigma, t) \neq 0 \quad (\sigma \neq 1/2). \quad (5.6)$$

Proof. Set

$$M(\sigma, t) = L_\varepsilon \operatorname{Re} D_T(\sigma + it). \quad (5.7)$$

Since $L_\varepsilon > 0$, the sign condition (5.1) gives

$$\operatorname{sgn}(M(\sigma, t)) = \operatorname{sgn}\left(\sigma - \frac{1}{2}\right), \quad M(\sigma, t) \neq 0 \quad (5.8)$$

whenever $\sigma \neq 1/2$. The domination estimate (5.2) becomes

$$|R_\varepsilon(\sigma, t)| \leq \theta|M(\sigma, t)|, \quad 0 < \theta < 1. \quad (5.9)$$

If $M > 0$, then

$$\mathcal{B}_\varepsilon = M + R_\varepsilon \geq M - |R_\varepsilon| \geq (1 - \theta)M > 0. \quad (5.10)$$

If $M < 0$, then

$$\mathcal{B}_\varepsilon = M + R_\varepsilon \leq M + |R_\varepsilon| \leq M - \theta M = (1 - \theta)M < 0. \quad (5.11)$$

Thus B_ε has the same sign as M , proving (5.5) and (5.6).

The quantitative content of the sign barrier is shown in **Figure 2**. With $M = M(\sigma, t)$ as in (5.7), equations (4.2) and (5.9) give the interval bound

$$M - \theta|M| \leq \mathcal{B}_\varepsilon(\sigma, t) \leq M + \theta|M|. \quad (5.12)$$

For $\theta = 0.4$, this interval has endpoints $0.6M$ and $1.4M$. Under the analytic sign condition, M is nonzero away from the critical line, and the whole interval has the sign of M . The bridge therefore retains at least 60% of the main term in absolute value and cannot cross zero away from the critical line.

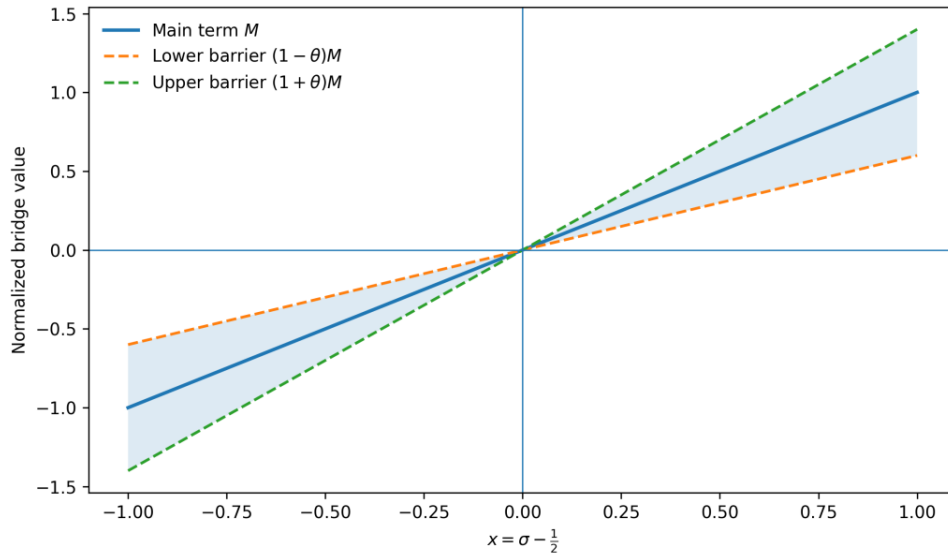


Figure 2: Normalized sign barrier for Proposition 27, shown for $\theta = 0.4$.

Zero-Adaptation and the Critical-Line Criterion

Let

$$\mathcal{Z} := \{\rho \in \mathbb{C} : \xi(\rho) = 0\} \quad (6.1)$$

denote the set of nontrivial zeros of ζ . Each zero can be written as

$$\rho = \sigma + it, \quad 0 < \sigma < 1. \quad (6.2)$$

Definition 28 (Zero-adapted bridge). A bridge $\mathcal{B}_\varepsilon$ is zero-adapted if

$$\mathcal{B}_\varepsilon(\sigma, t) = 0 \quad (6.3)$$

for every nontrivial zero $\rho = \sigma + it$ of ζ .

Assumption 29 (Bridge criterion data). A bridge datum $(T, \varepsilon, R_\varepsilon, \theta)$ is admissible for the criterion when all of the following conditions hold:

1. T is real-admissible;
2. $0 < \varepsilon < 1$;
3. $0 < \theta < 1$;
4. $\mathcal{B}_\varepsilon$ is the real bridge defined by (4.2);
5. T satisfies the analytic sign condition (5.1);
6. R_ε satisfies the domination estimate (5.2);
7. $\mathcal{B}_\varepsilon$ is zero-adapted in the sense of (6.3).

Theorem 30 (Conditional critical-line criterion). Let $(T, \varepsilon, R_\varepsilon, \theta)$ be admissible for the criterion in the sense of Assumption 29. Then every nontrivial zero ρ of ζ satisfies

$$\operatorname{Re}(\rho) = \frac{1}{2}.$$

Proof. Let $\rho = \sigma + it$ be a nontrivial zero. Zero-adaptation gives

$$\mathcal{B}_\varepsilon(\sigma, t) = 0. \quad (6.4)$$

If $\sigma \neq 1/2$, Proposition 27 gives

$$\mathcal{B}_\varepsilon(\sigma, t) \neq 0, \quad (6.5)$$

a contradiction. Hence $\sigma = 1/2$. Since ρ was arbitrary, every nontrivial zero lies on the critical line.

Corollary 31 (Equivalent formulation under critical normalization). Assume conditions 1-6 of Assumption 29 and

suppose that the bridge is critically normalized. Then the bridge is zero-adapted if and only if every nontrivial zero of ζ lies on the critical line.

Proof. If the bridge is zero-adapted, condition 7 is also satisfied, so Theorem 30 gives the critical-line conclusion. Conversely, suppose every nontrivial zero has the form $\rho = 1/2 + it$. Critical normalization and Proposition 19 give $\mathcal{B}_\varepsilon(1/2, t) = 0$ at each such zero. Hence the bridge is zero-adapted.

Remark 32. The theorem is a conditional implication. The reflected decomposition alone does not supply the analytic sign law, the domination estimate, or the zero-adapted identity. Corollary 31 makes the logical burden explicit: under critical normalization and the sign barrier, zero-adaptation is equivalent to the critical-line conclusion.

Table 2 summarizes the logical structure. The first three rows are proved within the paper, the next three are application-level requirements, and the last row is the consequence obtained from them. This separation prevents structural identities from being mistaken for the unresolved coercive input.

Ingredient	Mathematical content	Role in the argument
Reflected datum	Equations (2.4), (2.7), and (2.8)	Sets the analytic framework and defines the defect.
Real critical-line symmetry	Equations (2.18) and (2.19)	Identifies the real quantity that vanishes on the critical line.
Arithmetic scale	Proposition 12, especially (3.13)	Supplies the positive weight in the bridge.
Analytic sign law	Condition (5.1) off the critical line	Forces the main term to point away from zero off the line.
Correction bound	Estimate (5.2), with $0 < \theta < 1$	Prevents the correction from changing the sign of the bridge.
Zero-adaptation	Identity (6.3) at every nontrivial zero	Connects the bridge to the zero set of ξ .
Critical-line conclusion	Theorem 30: $\text{Re}(\rho)=1/2$ for every nontrivial zero	Follows from zero-adaptation and the sign barrier.

Table 2: Logical Structure of the Conditional Bridge Criterion

What Remains for an Application

The formal implication in Theorem 30 is complete. A substantive application, however, must produce the hypotheses from a construction that does not already assume the location of the zeros. Four ingredients are required.

- **A canonical reflected datum.** Proposition 4 shows that reflected data are abundant: every analytic function H with $H(1-s) = -H(s)$ gives $T = \xi/2 + H$. An application must select T from a specific Ramanujan-type representation of ξ or Ξ , rather than from an arbitrary odd perturbation.
- **Analytic coercivity.** The selected datum must satisfy the sign law (5.1) throughout the critical strip. Oddness about the critical line gives symmetry only; it does not give positivity on one side and negativity on the other. Example 8 shows that a sign law can be arranged for an arbitrary datum, but that example is not yet linked to Ramanujan's arithmetic structure or to the zero set.
- **A controlled correction.** The correction R_ε must arise from the same analytic-arithmetic construction as the bridge and must satisfy the strict domination estimate (5.2). This estimate is what prevents cancellation of the main term.
- **A derived zero-adaptation identity.** Equation (6.3) must be proved at every nontrivial zero from the chosen construction. It cannot be imposed by defining the correction from the unknown zero locations, because that would make the criterion circular.

A proposed construction should therefore be checked in the following order. First define T , R_ε , and ε independently of the zero set. Next prove real-admissibility, compute $D_{T,\varepsilon}$, and study the normalized real defect $A_{T,\varepsilon}$. The analytic sign condition is equivalent to $A_{T,\varepsilon}(\sigma, t) > 0$ off the critical line. Then choose the arithmetic scale; Proposition 12, specifically (3.13), gives $L_\varepsilon > 0$ for $0 < \varepsilon < 1$. After that, prove the correction estimate and finally derive zero-adaptation at the zeros of ξ .

This order keeps the unresolved content visible. The theorem does not assume the Riemann Hypothesis directly, but Corollary 31 shows that, under critical normalization and the sign barrier, zero-adaptation has the same force as the critical-line conclusion. A successful application must therefore obtain zero-adaptation from independent analytic and arithmetic identities, not from the desired conclusion

Conclusion

The bridge must be built from the real defect $\text{Re}D_T(\sigma + it)$ because the full complex defect D_T need not vanish on the critical line. Proposition 6 proves the reflection identity for the real defect and its vanishing on $\sigma = 1/2$. Proposition 12 supplies the positive arithmetic weight L_ε for $0 < \varepsilon < 1$. Propositions 26 and 27 then show that a correction satisfying the strict domination estimate cannot cancel or reverse the main term away from the critical line. Theorem 30 completes the logical mechanism: zero-adaptation at every nontrivial zero, together with the analytic sign law and the dominated correction, forces every such zero onto the critical line. The proof of this implication contains no omitted step. Its hypotheses remain the substantive mathematical burden. The proved part of the paper consists of the classification of reflected data, the real critical-line symmetry of the defect, the existence and positivity of the arithmetic scale, and the quantitative sign-transfer proposition. A complete application still requires a canonical Ramanujan-type datum, an analytic sign law for that datum, a correction bound, and a zero-adaptation identity derived independently of the zero locations. The paper therefore establishes a complete conditional criterion and does not establish the Riemann Hypothesis itself.

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