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Goldbach Entropy and Analytic Compression: A Structural Theory of Additive Arithmetic

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Abstract
This paper develops a foundational framework for understanding the structural limitations of analytic methods in additive number theory, using the binary Goldbach problem as a case study. We formalize how generating functions, Dirichlet series, circle-method asymptotics, and sieve bounds encode arithmetic information through averaging and smoothing (analytic compression). We define Goldbach entropy as a local invariant capturing representation structure, reformulate Goldbach as a uniform non-vanishing condition, and establish impossibility results explaining why average control cannot certify universal predicates.

Introduction
The binary Goldbach conjecture asserts that every even integer $N \geq 4$ can be written as a sum of two prime numbers. Despite its simple formulation, the conjecture has resisted proof for nearly three centuries. This resistance persists even though analytic number theory provides overwhelming statistical evidence in its favor, and computation has verified the conjecture for all even integers up to extremely large bounds.

This paper does not propose a new proof strategy for Goldbach, nor does it present new numerical bounds. Instead, it addresses a foundational and methodological question:
Why do analytic methods in number theory provide strong statistical evidence for Goldbach-type statements while appearing structurally incapable of certifying their universal truth?

The central thesis is that the difficulty of Goldbach is not primarily quantitative, but structural. Goldbach is a universal non-vanishing statement, whereas analytic methods are intrinsically designed to control averages, densities, and smoothed quantities. This average–minimum mismatch constitutes a genuine mathematical obstruction rather than a temporary limitation of technique.

Historical Background
The Hardy–Littlewood circle method predicts a precise asymptotic formula for the number of representations of a large even integer as a sum of two primes. Sieve methods show that "almost all" integers satisfy related additive properties, often under relaxed primality conditions. Probabilistic models of the primes suggest that Goldbach should hold with overwhelming likelihood, and large-scale computation confirms its truth for all integers within current computational reach.

Nevertheless, none of these approaches yields a proof that applies to every even integer. This persistent gap between evidence and proof motivates a closer examination of the logical and structural capabilities of analytic methods themselves.

Scope and Non-Claims
It is important to delineate clearly what this paper does and does not claim.
- We do *not* prove the Goldbach conjecture.
- We do *not* derive new bounds for prime representation functions.
- We do *not* propose an analytic continuation of arithmetic predicates.

Instead, the contribution is foundational. We provide a conceptual framework that explains why analytic approaches succeed at producing strong statistical evidence while remaining structurally unsuited for universal certification of additive predicates.

Goldbach as a Predicate and as Structure

Goldbach as a Logical Predicate

Let P denote the set of prime numbers. The binary Goldbach conjecture can be expressed as a predicate

$$AG(N) := 1\{\exists p, q \in P \text{ such that } p + q = N\},$$

defined on the even integers N ≥ 4.

This formulation emphasizes that Goldbach is not a numerical function but a logical predicate assigning a binary truth value to each integer. As such, it is inherently discrete and discontinuous: an infinitesimal perturbation of N has no meaning, and truth at one integer conveys no immediate information about truth at another.

Representation Functions and Non-Vanishing

Define the (ordered) Goldbach representation function

$$rG(N) := \#\{(p, q) \in P\times P : p + q = N\}.$$

Then Goldbach is equivalent to the statement that

$$rG(N) \geq 1 \text{ for all even } N \geq 4.$$

This formulation reveals an essential feature: Goldbach is a *non-vanishing* condition. It does not ask whether $rG(N)$ is large on average, but whether it ever vanishes. This distinction will play a central role throughout the paper.

Statistical Versus Universal Statements

Statements accessible to analytic number theory often take the form

$$\frac{1}{X} \sum_{\substack{N \leq X \\ N \text{ even}}} r_G(N) \sim (\text{predicted main term}),$$

or involve smoothed variants of such averages. These are statistical statements about typical behavior.
Goldbach, by contrast, is a universal statement:

$$\inf_{\substack{N \geq 4 \\ N \text{ even}}} r_G(N) \geq 1.$$

This is a statement about the minimum of a nonnegative arithmetic function, not its average. The logical strength required to certify a minimum is strictly greater than that required to control averages, and this gap is the source of the structural difficulty examined in this work.

From Counting to Structure

Beyond Raw Counts

While $rG(N)$ captures the number of representations, it discards information about how those representations are distributed. For example, two integers may have the same number of representations, but in one case the representations may be spread across many primes, while in another they may be highly concentrated.

This observation motivates a shift in perspective: from counting representations to studying the internal structure of the representation set.

Representation Structure

For a fixed even integer N, define the representation support

$$S_G(N) := \{p \in P : N - p \in P\}.$$

This set records which primes participate in representations of N.
The cardinality of $S_G(N)$ determines whether Goldbach holds at N, but the distribution of $S_G(N)$ across the prime scale carries additional structural information. This motivates the introduction of entropy-like invariants in later sections.

Preview of the Structural Framework

The remainder of the paper develops three interconnected ideas:
- Analytic methods act as *compression operators* that preserve global statistics while discarding pointwise structure.
- Goldbach can be reformulated exactly as a uniform positivity condition on a local structural invariant (Goldbach entropy).
- No method based purely on averaged or compressed information can certify such a uniform condition.

Together, these ideas provide a structural explanation for the long-standing gap between analytic evidence and universal proof in additive number theory.

Analytic Compression

Analytic number theory studies arithmetic objects by embedding them into continuous or analytic structures. This embedding enables the use of powerful tools from Fourier analysis, complex analysis, and harmonic analysis. At the same time, it fundamentally changes the nature of the information being studied. In this section we formalize this change as analytic compression.

Conceptual Description

Analytic methods rarely work with exact arithmetic data. Instead, they replace discrete information with averaged, smoothed, or transformed surrogates. This process suppresses local irregularities while amplifying global structure. Compression is therefore not a defect of analytic methods; it is the source of their effectiveness.

However, compression inevitably discards pointwise information. As a result, analytic methods are naturally aligned with statistical statements rather than universal logical predicates.

Standard Examples of Analytic Compression

We list several classical constructions that exemplify analytic compression.

Generating Functions

Given an arithmetic sequence (an), the generating function

$$F(z) = \sum_{n=1}^{\infty} a_n z^n$$

encodes infinitely many discrete values into a single analytic object. While individual coefficients can be recovered in principle, most analytic arguments rely on global properties such as convergence, singularities, or average growth, not exact pointwise values.

Dirichlet Series Dirichlet series of the form

$$D(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

encode arithmetic data through a complex variable. Analytic continuation and mean-value estimates reveal global behavior, but exact values of an are not directly accessible from the compressed analytic object.

The Hardy–Littlewood Circle Method

The Hardy–Littlewood circle method expresses additive representation counts as integrals of exponential sums over the unit circle. Major arcs capture structured average behavior, while minor arcs are controlled only in aggregate. The method is therefore intrinsically designed to estimate typical size rather than certify pointwise non-vanishing.

Sieve Methods

Sieve theory eliminates integers failing certain congruence conditions. While sieves can bound exceptional sets and prove "almost all" results, they typically collapse exact primality conditions into coarse exclusion statistics. This reflects another form of compression: exact arithmetic structure is replaced by approximate filtering.

Finite Perturbation Invariance

A key feature of analytic compression is its insensitivity to finite changes in the input data.
Definition (Finite Perturbation Invariance). An analytic compression is said to be finiteperturbation invariant if modifying the underlying arithmetic data at finitely many integers does not change the compressed output.

Asymptotic densities, mean values, smoothed sums, and analytic continuations typically satisfy this property. This invariance will play a crucial role in later impossibility results.

An Operator Framework for Additive Arithmetic

We now introduce a minimal operator framework that separates three logically distinct steps often conflated in informal discussions: forming arithmetic predicates, compressing arithmetic data analytically, and attempting to recover predicates from compressed information.

Arithmetic Predicate Operator

Definition (Arithmetic Predicate Operator). Let $S \subset \mathbb{Z}$ be a discrete set. The arithmetic predicate operator of arity two is defined by

$$\mathrm{APO}_2(S)(n) := \mathbf{1}\{\exists a, b \in S \text{ such that } a + b = n\}.$$

For $S = \mathbb{P}$, the set of primes, this operator recovers the Goldbach predicate. The operator APO_2 acts directly on discrete arithmetic structure and produces a logical truth value.

Analytic Compression Operator

Definition (Analytic Compression Operator). An analytic compression operator is any mapping that sends arithmetic data to an analytic or continuous surrogate while preserving global statistics but generally losing exact pointwise information.

Generating functions, Dirichlet series, circle-method integrals, and sieve bounds all fall under this definition. Non-injectivity is essential: distinct arithmetic inputs may yield identical compressed outputs.

Predicate Projection Operator

Definition (Predicate Projection Operator). A predicate projection operator is any attempted rule that maps compressed analytic data back to a pointwise arithmetic predicate.

The central methodological problem addressed in this paper is the failure of predicate projection to recover universal arithmetic truth from compressed information.

Order Sensitivity

Arithmetic predicate formation and analytic compression are order-sensitive operations. Applying compression before predicate evaluation generally destroys the discrete structure required for correct logical certification. This is not algebraic non-commutativity, but an irreversible loss of information.

Structural Principles of Arithmetic–Analytic Interaction

Remark The following principles summarize recurring structural features of analytic number theory. They are methodological observations rather than axioms of mathematics.

Principle (Discreteness)
Arithmetic predicates depend on exact membership in discrete sets and are not continuous under perturbation.

Principle (Compression)
Analytic methods encode arithmetic data through averaging, smoothing, or transformation into continuous objects.

Principle (Finite Invisibility)
Analytic compressions are typically invariant under finite modification of the input data.

Principle (Local Certification)
Universal arithmetic predicates require pointwise non-vanishing of a local structural invariant.

Goldbach Entropy

This section introduces a local structural invariant, called *Goldbach entropy,* which reformulates the Goldbach predicate as a uniform non-vanishing condition. The purpose is to replace a binary logical statement with a quantitative object that retains exact logical equivalence while allowing finer structural analysis.

Representation Support

For an even integer $N \geq 4$, define the Goldbach representation support

$$S_G(N) := \{p \in P : N - p \in P\}.$$

The set $S_G(N)$ records which primes participate in representations of N as a sum of two primes.
By construction, $S_G(N)$ is nonempty if and only if N satisfies the Goldbach property. Thus the Goldbach conjecture is equivalent to the assertion that $SG(N)$ is nonempty for all even $N \geq 4$.

Support Entropy

Definition (Support entropy). Define the support entropy of N by

$$H_G^{\mathrm{supp}}(N) := \log\bigl(1 + |\mathcal{S}_G(N)|\bigr).$$

The logarithm is introduced for normalization and monotonicity; the addition of 1 ensures the quantity is defined for all N and vanishes precisely when $S_G(N)$ is empty.

Theorem (Exact reformulation of Goldbach). *For even integers $N \geq 4$, the following are equivalent:*

- *N satisfies the Goldbach conjecture;*
- $S_G(N) \neq \emptyset$;
- $HG\mathrm{supp}(N) > 0$.

Proof. The equivalence of (1) and (2) follows directly from the definition of the Goldbach predicate. Condition (3) holds if and only if $1 + |S_G(N)| > 1$, which is equivalent to $|S_G(N)| \geq 1$, i.e. $SG(N)$ is nonempty.

Remark Theorem 7.1 is elementary but conceptually powerful. It converts Goldbach from a logical predicate into a minimum problem: the conjecture asserts that a nonnegative local invariant never vanishes.

Goldbach as a Minimum-Type Statement

The support-entropy formulation highlights a structural distinction between Goldbach and the types of statements typically accessible to analytic methods. Analytic arguments naturally estimate averages such as

$$\frac{1}{X}\sum_{N\leq X}|\mathcal{S}_G(N)|,$$

or smoothed variants thereof. Goldbach, however, requires the much stronger statement

$$\inf_{\substack{N\geq 4\\ N\ \text{even}}} H_G^{\text{supp}}(N)>0.$$

This difference between average control and minimum control underlies the obstruction analyzed in subsequent sections.

Weighted Goldbach Entropy

Support entropy detects existence but does not distinguish between sparse and rich representation structure. To connect with classical analytic heuristics, we introduce a weighted refinement compatible with Hardy–Littlewood-type weights.

Definition (Weighted representation weights). Fix even $N \geq 4$. For primes $p \leq N-2$, define

$$w_N(p):=\begin{cases}\log p\cdot\log(N-p), & \text{if } p\in\mathcal{P} \text{ and } N-p\in\mathcal{P},\\ 0, & \text{otherwise.}\end{cases}$$

Let $W(N):=\sum_{p\leq N-2} w_N(p)$.

When $W(N) > 0$, define a probability mass function

$$P_N(p):=\frac{w_N(p)}{W(N)}.$$

Definition (Weighted Goldbach entropy). The *weighted Goldbach entropy* is defined by

$$H_G^{\text{wt}}(N):=\begin{cases}-\sum_{p\leq N-2}P_N(p)\log P_N(p), & \text{if } W(N)>0,\\ 0, & \text{if } W(N)=0.\end{cases}$$

Remark Weighted entropy measures how dispersed the representation mass is across primes. Support entropy is logically exact; weighted entropy is a refinement that captures additional structural information but does not alter the minimum-type nature of the Goldbach predicate.

Interpretation

From the entropy perspective, analytic methods typically predict that $H_G^{\text{wt}}(N)$ is large on average and grows with N. Goldbach, however, requires that the coarser invariant $H_G^{\text{supp}}(N)$ never vanishes. The next section formalizes why compression-based analytic information cannot, by itself, guarantee this uniform non-vanishing property.

Structural No-Go Theorems

This section establishes two general obstructions explaining why analytic information obtained through compression cannot, by itself, certify universal non-vanishing of a local invariant such as Goldbach entropy. These obstructions are independent of the specific arithmetic problem and arise from basic features of averaging and information loss.

The Average–Minimum Barrier

We begin with an obstruction that is purely analytic in nature. It formalizes the difference between controlling the average of a nonnegative function and controlling its minimum.

Theorem (Average–minimum barrier). *Let $f:\mathbb{N}\to[0,\infty)$ be any nonnegative function. Knowledge that the averages*

$$\frac{1}{X}\sum_{n\leq X}f(n)$$

are uniformly large as $X\to\infty$ does not imply the existence of a uniform positive lower bound on f. In particular, large average Goldbach entropy does not imply pointwise positivity at every even integer.

Proof. Fix any target average level $M > 0$. Define $f(n) = 0$ on a sparse infinite set, for example on perfect squares, and define $f(n)$ to be sufficiently large on the complement so that the average over $n \leq X$ exceeds M for all sufficiently large X. Then $\inf_n f(n) = 0$ even though the average is large. This shows that average control cannot exclude isolated zeros.

Remark Theorem 8.1 is a manifestation of a general analytic principle: L^1-type information does not control L^∞ behavior from below. In the context of Goldbach, this explains why asymptotic formulas and mean-value estimates do not automatically imply universal representability.

Consequences for Goldbach Entropy

Applying Theorem 8.1 to the nonnegative function $f(N) = H_G^{\text{supp}}(N)$ shows that control of average support entropy cannot rule out isolated values of N for which $H_G^{\text{supp}}(N) = 0$. Thus even very strong analytic evidence is logically compatible with finitely many Goldbach failures.

Finite Perturbation Invisibility

The second obstruction is information-theoretic. Many analytic compressions are insensitive to finite modifications of the underlying arithmetic data.

Definition (Finite perturbation invariance). An analytic compression C is said to be finiteperturbation invariant if C(X) = C(X') whenever the arithmetic data X and X' differ on only finitely many inputs.
1
Examples include asymptotic densities, mean values, smoothed sums, and analytic continuations of Dirichlet series.

Theorem (Finite perturbation invisibility). Let C be a finite-perturbation invariant analytic compression, and let Π be any decision rule that attempts to certify a universal arithmetic predicate using only C(X). Then Π cannot rule out predicates that may fail at finitely many points. In particular, compression alone cannot exclude finitely many Goldbach counterexamples.
Proof. Suppose X is arithmetic data for which the predicate holds universally. Modify X on finitely many carefully chosen inputs to obtain X' for which the predicate fails at those inputs. By finite-perturbation invariance, C(X) = C(X'). Any decision rule Π depending only on C(X) must therefore give the same output for X and X', and hence cannot be universally correct.

Corollary No approach relying exclusively on compression-invariant analytic information can, by itself, certify the Goldbach conjecture in its universal form.

Remark Corollary 8.1 does not assert that Goldbach is false or that analytic methods are ineffective. It identifies a structural limitation: any proof of Goldbach must either evade pure compression invariance or introduce a mechanism that upgrades average control to genuine minimum control.

Methodological Interpretation

Together, the average–minimum barrier and finite perturbation invisibility explain why analytic methods naturally produce strong evidence without yielding universal certification. They formalize the intuitive gap between "almost all" statements and "all" statements and show that the gap is not merely technical but structural.
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A Hierarchy of Analytic Power

To understand precisely where analytic methods succeed and where they necessarily stop, we classify the strength of conclusions they can deliver. The hierarchy below is expressed in terms of nonnegative arithmetic invariants, such as representation counts or entropy-like quantities.

Levels of Control

Definition (Analytic power levels). Let $F(N) \geq 0$ be an arithmetic invariant. We say a method achieves:

- *Level I control* if it establishes global mean estimates for

$$\frac{1}{X}\sum_{N\leq X} F(N).$$

- *Level II control* if it establishes smoothed local estimates of the form

$$\sum_{N} F(N)\, w(N/X)$$

for suitable smoothing weights w.

- *Level III control* if it proves $F(N) > 0$ for all but $o(X)$ values of $N \leq X$ (exceptional set bounds).

- *Level IV control* if it proves $F(N) > 0$ for all relevant N (uniform pointwise positivity).

This hierarchy isolates the logical jump between statements about "almost all" integers and statements about "all" integers.

Goldbach and Level IV

Proposition The Goldbach conjecture is equivalent to Level IV control for the support entropy $H_G^{supp}(N)$ over even integers $N \geq 4$.

Proof. Recall that the Goldbach support set is defined by

$$S_G(N) = \{p \in P : N - p \in P\},$$

and the support entropy by

$$H_G^{\text{supp}}(N) = \log\bigl(1 + |\mathcal{S}_G(N)|\bigr).$$

Goldbach holds at N if and only if $S_G(N) \neq \emptyset$, i.e. if and only if $|S_G(N)| \geq 1$. Since $\log(1 + t) > 0$ precisely when $t > 0$, this condition is equivalent to $H_G^{supp}(N) > 0$.

Therefore the universal Goldbach statement

"for all even $N \geq 4$, $A_G(N) = 1$"

is exactly the uniform positivity statement

$$H_G^{\text{supp}}(N) > 0 \quad \text{for all even } N \geq 4,$$

which is Level IV control in the hierarchy.

Remark The significance of Proposition 9.1 is not technical but structural: it shows that Goldbach is a *minimum-type* statement rather than an average-type one.

Circle Method and Sieve Theory Through the Entropy Lens

The Circle Method

The Hardy–Littlewood circle method expresses additive representation functions as Fourier-analytic integrals. Major arcs encode arithmetic structure and yield main terms for averaged or smoothed statistics, while minor arcs are controlled in aggregate norms. Consequently, the circle method naturally produces Level II control.

From the entropy perspective, the circle method predicts that weighted Goldbach entropy is large on average and increases with N. However, it does not provide a mechanism that prevents small or zero entropy at isolated integers. Thus it does not directly address the Level IV requirement.

Sieve Methods

Sieve methods eliminate candidates violating local congruence constraints. Modern sieve techniques can bound exceptional sets and thereby often achieve Level III control for related additive problems. However, classical sieve methods encounter parity-type obstructions that prevent simultaneous certification of primality in multiple positions.

In entropy language, sieve methods preserve average representation mass but do not automatically preserve a uniform entropy floor.

Remark The entropy framework does not replace classical tools. It clarifies their native output type and explains why additional mechanisms are required to pass from Level III to Level IV.

Model Obstruction and Indistinguishability

We now formalize a structural phenomenon showing that compression-based evidence can be logically insufficient even when it is arbitrarily strong.

Compression Indistinguishability

Definition (Compression indistinguishability). Two arithmetic worlds X and X' are said to be *indistinguishable* under an analytic compression C if $C(X) = C(X')$.

Finite Forcing Lemma

Lemma (Finite forcing lemma). *Let* $x : \mathbb{N} \to \{0,1\}$ *and define the additive predicate*

$$A_x(N) = 1\{\exists a + b = N \text{ with } x(a) = x(b) = 1\}.$$

For any finite set of even integers $E \subset 2\mathbb{N}$, there exists a function x' differing from x on only finitely many inputs such that $A_x'(N) = 0$ for all $N \in E$.

Proof. Fix $N \in E$. The set of representations $a + b = N$ with $x(a) = x(b) = 1$ is finite. Remove membership (set value to 0) for at least one endpoint from each such pair. Repeating this procedure for all $N \in E$ modifies x on only finitely many inputs and ensures $A_x'(N) = 0$ for all $N \in E$.

Model Obstruction Theorem

Theorem (Model obstruction theorem). There exist arithmetic models X and X′ such that:

- X and X′ are indistinguishable under broad classes of analytic compressions that capture averaged and smoothed statistics; yet
- the Goldbach-type predicate differs at finitely many integers between X and X′.

Consequently, compression-based analytic evidence cannot logically exclude sparse predicate failure.

Proof. Let x encode an arithmetic world and let C be a finite-perturbation invariant compression. Choose a finite set of even integers E at which predicate failure is desired. By Lemma 11.1, there exists x' differing from x on finitely many inputs such that $A_x'(N) = 0$ for all $N \in E$.

Since x and x' differ only on finitely many inputs and C is invariant under such perturbations, we have $C(x) = C(x')$. Thus x and x' are indistinguishable by C even though their induced Goldbach-type predicates differ on E. This establishes the claimed obstruction.

Remark The theorem does not assert that the primes themselves exhibit such behavior. It shows that no amount of compression-based evidence can, by itself, rule out sparse failures without additional non-compressive input.

Summary

The hierarchy of analytic power and the model obstruction theorem together explain why analytic methods can produce compelling evidence and strong partial results without delivering universal certification. Any proof of Goldbach must therefore overcome the Level III to Level IV barrier by introducing a genuinely new mechanism.

Generalization Beyond Goldbach

The framework developed in this paper is not specific to the binary Goldbach conjecture. Many additive problems in number theory can be expressed as universal non-vanishing predicates derived from a discrete set. In each case, the predicate can be reformulated as uniform positivity of a local structural invariant.

Additive Predicates from Discrete Sets

Let $S \subset Z$ be a discrete set and let $k \geq 2$. Define the additive predicate

$$P_{S,k}(n) := 1\{\exists a1,\dots,a_k \in S \text{ such that } a1 + \cdots + a_k = n\}.$$

Associated to this predicate is a support set consisting of partial representations, whose cardinality measures local representability. A support entropy

$$H_{S,k}(n) := \log 1 + \text{support size at } n$$

vanishes exactly when the predicate fails.

Remark Prime k-tuple conjectures, additive problems involving squares or higher powers, and many combinatorial representation problems fit this template. In each case, analytic methods control averaged correlations, while the conjecture requires uniform non-vanishing.

A Research Program Toward Entropy-Sensitive Methods

The results above identify a precise obstruction: analytic compression controls averages but not minima. Any future breakthrough must therefore either evade compression invariance or introduce a mechanism that upgrades average control to uniform positivity.

Representation-Space Sieve

A promising direction is to move from sieving integers to sieving *representation spaces*. For fixed even N, Goldbach representations correspond to candidates p with $q = N - p$. One may sieve the candidate set by excluding values of p and q divisible by small primes.

The entropy-sensitive perspective tracks not only how many candidates survive, but whether the surviving candidates remain sufficiently dispersed across residue classes and scales as the sieve depth increases.

Dispersion Stability

A prototype target statement for an entropy-sensitive method is the following:
After successive local exclusions, the surviving candidate set retains a uniform dispersion lower bound independent of N. Such a statement would prevent entropy collapse and could potentially bridge the gap between average information and uniform certification.

Remark Current sieve and harmonic methods are effective at preserving average mass but do not automatically preserve dispersion floors. Establishing such floors would require new ideas beyond classical compression-based techniques.

Falsifiability and Structural Boundaries

A foundational framework gains strength when it makes falsifiable claims. The present framework would be undermined by any method that achieves universal certification while respecting the compression constraints identified earlier.

Proposition (Falsifiability condition). The core obstruction results of this paper would be refuted by a method that:
- relies exclusively on compression-invariant analytic information, and

• nevertheless certifies a universal non-vanishing predicate equivalent to Goldbach.\

Remark The proposition does not assert that such a method cannot exist. It asserts that if it exists, it must violate at least one structural feature identified here, such as pure compression invariance or purely averaged reasoning.

Conclusion

We have developed a structural theory of the arithmetic–analytic boundary using the Goldbach conjecture as a case study. By introducing Goldbach entropy, we reformulated Goldbach as a uniform positivity condition on a local invariant. Analytic compression was shown to preserve averages while discarding precisely the information required for minimum-type certification.

The resulting no-go theorems explain why analytic number theory produces overwhelming evidence without yielding universal proofs. The hierarchy of analytic power clarifies where classical methods reside and what kind of new mechanism a future breakthrough would need to introduce.

The value of the framework is explanatory and organizational. It does not solve Goldbach, but it clarifies why solving it requires fundamentally new control beyond existing analytic techniques.

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