Journal of Theoretical, Experimental, and Applied Physics

Volume 1, Issue 1

Research Article

Date of Submission: 10 Oct, 2025
Date of Acceptance: 31 Oct, 2025
Date of Publication: 10 Nov, 2025

Lie Symmetries and Conserved Quantities of Four-Dimensional Static Spacetime in Spherical Coordinate Systems

Sun Xiao Fan[1*], Jing Li Fu[1,2], Yong Xin Guo[2] and Hui Dong Cheng[1]

[1]School of Artificial Intelligence, Shandong Vocational University of Foreign Affairs, China

[2]Department of Physics, Liaoning University, China

[3]Department of Physics, Zhejiang Sci-Tech University, China

***Corresponding Author:**
Sun Xiao Fan, School of Artificial Intelligence, Shandong Vocational University of Foreign Affairs, China.

Citation: Sun, X. F., Fu, J. L.., Y. X., Cheng, H. D. (2025). Lie Symmetries and Conserved Quantities of Four-Dimensional Static Spacetime in Spherical Coordinate Systems. *J Theor Exp Appl Phys, 1*(1), 01-09.

Abstract
In this study, the Lagrange equation for static four-dimensional spacetime in spherical coordinate system was presented; Introducing the Lie group of transformations about four-dimensional static spacetime, we constructed an infinitely small generator vector field and its first-order and second-order extensions for four-dimensional static spacetime; Based on the invariance of the Lagrange equation for four-dimensional static spacetime in spherical coordinate system under the transformation of Lie group, the Lie symmetry determination equation of the system and its corresponding Lie symmetry partial differential equation system are given; Solving a series of partial differential equations and obtaining a definite transformation Lie group; The Lie symmetry theorem for four-dimensional static spacetime in spherical coordinate system was proposed, and the conservation quantity (first integral) determined by the metric coefficient of the system was obtained; It has been discovered that any value of the metric coefficient in four-dimensional static spacetime in a spherical coordinate system corresponds to a conserved quantity, which means that there can be countless conserved quantities corresponding to the metric coefficient in four-dimensional static spacetime in a spherical coordinate system. This article systematically establishes the Lie symmetry theory of four-dimensional static spacetime in spherical coordinate system.

Keywords: 4-Dimensional Spacetime, Lie Symmetry and Conserved Quantity, Lagrange Equation, Spherical Coordinate System

Introduction
As it is generally known, spacetime symmetries have a significant role in the motion of particles specifically in gravity theories. The classification of spacetime symmetry has become a hot topic in this study, the Lie group analysis is applied to 4-dimensional spacetime in spherical coordination systems and to give classifications of the systems with Lie symmetries. Firstly, the Lagrange equations of static space-time in spherical coordination systems are established. Secondly, Lie symmetry determining equations and Lie theorem of the static spacetime are studied. Thirdly, classifications of different class equations of systems are discussed along with their Lie symmetries and conservation laws. The main objective of this research is to obtain all possible forms of Lie symmetry classification for the static spacetime and discuss their physical significance.in general relativity [1-7]. As widely observed, a dynamical equation is in fact a differential equation that describe the motion of particles. In 1974, Lie introduced infinitesimal transformations into differential equations and proposed a symmetric solution for solving differential equations [8]. Afterwards, the symmetry method became a fundamental method for solving differential equations [9-15]. By using the symmetry solving method, we can solve dynamical equations; reduce the order of differential equations; and linearize nonlinear dynamical equations. These methods are used in the reduction of variables in partial differential equations. There are two basic symmetry methods under the transformation Lie group. One is based on the invariance of the Hamiltonian action of the dynamic system

under the transformation Lie group, which is called the Noether symmetry method; Other is based on the invariance of dynamical equations of system under the transformation Lie group, which is called the Lie symmetry method [8,16,17]. In recent years, the Lie symmetry method has been successfully applied to solve problems in conservative and non-conservative, holonomic and non-holonomic constrained mechanical systems, and phase space constrained mechanical systems [18-30]. Scholars have applied the Lie symmetry method to the problems of solving electromechanical coupled dynamic systems and flexible robot systems [31-36].

Apart from symmetries of dynamical equations, there are different types of spacetime symmetries which are used to classify different space-time [37-39]. This type of categorization of the space-time metrics not only provide different classes of space-time, but also provides the exact and new solutions of Einstein field equations,

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = k T_{\mu\nu} ,$$

where, $k = 8\pi G / c^4$. The categorization of spacetimes metrics comprises a crucial part of research of general theory of relativity [40,41]. Researchers of general relativity used different techniques to categorize the space-time metrics which are the exact solutions of Einstein field equations. The categorization of spherical, plane and cylindrical symmetric static and non-static metrics of spacetime related to different spacetime symmetries are discussed in the above-mentioned references. Noether symmetries play a significant role in classification of spacetime [42-44]. Generally, the B-R spacetime acknowledge SO (3) as the minimal isometry group. For static B-R spacetime, there is one isometry corresponding to the energy content of spacetime and one symmetry is the invariance of differential equations under the transformation Lie group, i.e. Lie symmetry.

In this article, we will investigate the Lie symmetry and conserved quantities of four-dimensional static spacetime in spherical coordination systems. The Lagrange equation of the system is obtained, and the transformation Lie group and determining equations of Lie symmetry for the system are derived. The Lie symmetry theorem and conservation quantity of the system are proposed., so that all possible conserved quantities of non-null electromagnetic field solution of the Einstein-Maxwell equations can be found.

Motion Equations of Four-Dimensional Static Spacetime in Spherical Coordinate System

In a 4-dimensional spacetime, generally the line element of nth dimensional spacetime takes the form:

$$ds^2 = g_{ij} dx^i dx^j \quad i, j = 1, 2, \cdots, n \tag{1}$$

The Lagrangian L for the metric (1) is given by [3–6]:

$$L = g_{ij} \dot{x}^i \dot{x}^j , \tag{2}$$

where, the g_{ij} is metric matrix. The usual Lagrangian L for a general symmetric four-dimensional static spacetime in spherical coordinate system is

$$L = \frac{1}{2}\left[e^{\nu(r)} \dot{t}^2 - e^{\mu(r)} \dot{r}^2 - e^{\lambda(r)} \left(\dot{\theta}^2 + \sin^2 \theta \dot{\varphi}^2 \right) \right] \tag{3}$$

here s is the independent variable, x^i are the dependent variables on t, r, θ, φ and $\dot{x}^i$ are their derivatives with respect to curve s; and here $\nu(r)$, $\mu(r)$ and $\lambda(r)$ are functions of radial coordinate r. The static spacetime in spherical coordinate system can be seen as a holonomic dynamical system that satisfies the Lagrange equation:

$$\frac{d}{ds} \frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} = Q_k \quad q_k = q_k \left(s, t, r, \theta, \varphi \right). \tag{4}$$

The Lagrange equations (4) is different from the Lagrange equation in analytical mechanics [48], as it is the Lagrange equations in four-dimensional spacetime.

Substituting Lagrangian (3) into equations (4), we obtain the Lagrange equations for the static spacetime in spherical coordinate system as follows:

$$\ddot{t} = -\frac{d\nu}{dr}\dot{r}\dot{t} = \alpha_0 , \tag{5}$$

$$\ddot{r} = -\frac{1}{2}\frac{d\mu}{dr}\dot{r}^2 - \frac{1}{2}e^{\nu(r)-\mu(r)}\frac{d\nu}{dr}\dot{t}^2 + \frac{1}{2}e^{\lambda(r)-\mu(r)}\frac{d\lambda}{dr}\left(\dot{\theta}^2 + \sin^2\theta\dot{\varphi}^2\right) = \alpha_1 , \tag{6}$$

$$\ddot{\theta} = -\frac{d\lambda}{dr}\dot{r}\dot{\theta} + \sin\theta\cos\theta\dot{\varphi}^2 = \alpha_2 , \tag{7}$$

$$\ddot{\varphi} = -\frac{d\lambda}{dr}\dot{r}\dot{\varphi} - 2\cot\theta\dot{\theta}\dot{\varphi} = \alpha_3 . \tag{8}$$

Lie Symmetries and Conserved Quantities of Four-Dimensional Static Spacetime in Spherical Coordinate System

Lie Symmetries and its Determining Equations of Four-Dimensional Static Spacetime

Introducing infinitesimal transformations on independent variable s and generalized coordinates,

$$s^* = s + \varepsilon\xi\left(s, q_0, q_1, \cdots, q_k\right), \quad q_k^* = q_k + \varepsilon\eta^k\left(s, q_0, q_1, \cdots, q_k\right) \quad k = 0,1,2,3 \tag{9}$$

where ε is an infinitesimal parameter, ξ and η^k are infinitesimal generators. Introducing the following infinitesimal generator vector,

$$X^{[0]} = \xi\frac{\partial}{\partial s} + \sum_{k=0}^{3}\eta^k\frac{\partial}{\partial q_k} , \tag{10}$$

and, one of its extensions,

$$X^{[1]} = X^{[0]} + \sum_{k=0}^{3}\left(\dot{\eta}^k - \dot{q}_k\dot{\xi}\right)\frac{\partial}{\partial \dot{q}_k} , \tag{11}$$

and its secondary expansion,

$$X^{[2]} = X^{[1]} + \sum_{k=1}^{4}\left(\ddot{\eta}^k - 2\alpha_k\dot{\xi} - \dot{q}_k\ddot{\xi}\right)\frac{\partial}{\partial \dot{q}_k} . \tag{12}$$

The invariance of differential equation (5~8) under infinitesimal transformation (9) is reduced to the following equation:

$$\ddot{\eta}^0 + 2\left(\frac{d\nu}{dr}\dot{r}\dot{t}\right)\dot{\xi} - \dot{t}\ddot{\xi} = -X^1\left(\frac{d\nu}{dr}\dot{r}\dot{t}\right), \tag{13}$$

$$\begin{aligned} &\ddot{\eta}^1 + \left(\frac{d\mu}{dr}\dot{r}^2 + e^{\nu(r)-\mu(r)}\frac{d\nu}{dr}\dot{t}^2 - e^{\lambda(r)-\mu(r)}\frac{d\lambda}{dr}\left(\dot{\theta}^2 + \sin^2\theta\dot{\varphi}^2\right)\right)\dot{\xi} - \dot{r}\ddot{\xi} \\ &= -X^{(1)}\left(\frac{1}{2}\frac{d\mu}{dr}\dot{r}^2 + \frac{1}{2}e^{\nu(r)-\mu(r)}\frac{d\nu}{dr}\dot{t}^2 - \frac{1}{2}e^{\lambda(r)-\mu(r)}\frac{d\lambda}{dr}\left(\dot{\theta}^2 + \sin^2\theta\dot{\varphi}^2\right)\right), \end{aligned} \tag{14}$$

$$\ddot{\eta}^2 + 2\left(\frac{d\lambda}{dr}\dot{r}\dot{\theta} - \sin\theta\cos\theta\dot{\varphi}^2\right)\dot{\xi} - \dot{\theta}\ddot{\xi} = -X^{(1)}\left(\frac{d\lambda}{dr}\dot{r}\dot{\theta} - \sin\theta\cos\theta\dot{\varphi}^2\right), \tag{15}$$

$$\ddot{\eta}^3 + 2\left(\frac{d\lambda}{dr}\dot{r}\dot{\varphi} + 2\cot\theta\dot{\theta}\dot{\varphi}\right)\dot{\xi} - \dot{\varphi}\ddot{\xi} = -X^{(1)}\left(\frac{d\lambda}{dr}\dot{r}\dot{\varphi} + 2\cot\theta\dot{\theta}\dot{\varphi}\right), \tag{16}$$

Equations (13)~(16) are called the Lie symmetry determination equations for the four-dimensional static spacetime in spherical coordinate system.

By expanding equations (13)~(16) respectively, we obtain the following partial differential equations of the four-dimensional static spacetime in spherical coordinate system:

$$\frac{\partial^2\eta^0}{\partial s^2}-\frac{\partial^2\xi}{\partial s^2}=0,\quad 2\frac{\partial^2\eta^0}{\partial t\partial s}-2\frac{\partial^2\xi}{\partial t\partial s}+\frac{\partial\eta^1}{\partial s}\frac{d\nu}{dr}=0,+2\frac{\partial^2\eta^0}{\partial\varphi\partial s}-2\frac{\partial^2\xi}{\partial\varphi\partial s}=0,$$

$$2\frac{\partial^2\eta^0}{\partial r\partial s}-2\frac{\partial^2\xi}{\partial r\partial s}+\frac{\partial\eta^0}{\partial s}\frac{d\nu}{dr}=0,\quad 2\frac{\partial^2\eta^0}{\partial\theta\partial s}-2\frac{\partial^2\xi}{\partial\theta\partial s}=0,$$

$$2\frac{\partial^2\eta^0}{\partial r\partial t}+2\frac{d\nu}{dr}\frac{\partial\xi}{\partial s}+\eta^1\frac{d^2\nu}{dr^2}+\frac{\partial\eta^1}{\partial r}\frac{d\nu}{dr}-2\frac{\partial\xi}{\partial s}\frac{d\nu}{dr}=0,$$

$$2\frac{\partial^2\eta^0}{\partial\theta\partial t}+\frac{\partial\eta^1}{\partial\theta}\frac{d\nu}{dr}=0,\quad 2\frac{\partial^2\eta^0}{\partial\varphi\partial t}+\frac{\partial\eta^1}{\partial\varphi}\frac{d\nu}{dr}=0,$$

$$2\frac{\partial^2\eta^0}{\partial\theta\partial r}-\frac{d\lambda}{dr}\frac{\partial\eta^0}{\partial\theta}+\frac{\partial\eta^0}{\partial\theta}\frac{d\nu}{dr}=0,\quad 2\frac{\partial^2\eta^0}{\partial\varphi\partial r}-\frac{d\lambda}{dr}\frac{\partial\eta^0}{\partial\varphi}+\frac{\partial\eta^0}{\partial\varphi}\frac{d\nu}{dr}=0,$$

$$2\frac{\partial^2\eta^0}{\partial\varphi\partial\theta}-2\cot\theta\frac{\partial\eta^0}{\partial\varphi}=0,\quad \frac{\partial^2\eta^0}{\partial r\partial r}-\frac{1}{2}\frac{d\mu}{dr}\frac{\partial\eta^0}{\partial r}+\frac{\partial\eta^0}{\partial r}\frac{d\nu}{dr}=0,$$

$$\frac{\partial^2\eta^0}{\partial t^2}-\frac{1}{2}e^{\nu(r)-\mu(r)}\frac{d\nu}{dr}\frac{\partial\eta^0}{\partial r}+\frac{\partial\eta^1}{\partial t}\frac{d\nu}{dr}=0,\quad \frac{\partial^2\eta^0}{\partial\theta\partial\theta}+\frac{1}{2}e^{\lambda(r)-\mu(r)}\frac{d\lambda}{dr}\frac{\partial\eta^0}{\partial r}=0,$$

$$\frac{\partial^2\eta^0}{\partial\varphi\partial\varphi}+\frac{1}{2}e^{\lambda(r)-\mu(r)}\frac{d\lambda}{dr}\frac{\partial\eta^0}{\partial r}\sin^2\theta+\sin\theta\cos\theta\frac{\partial\eta^0}{\partial\theta}=0,$$

$$-2\frac{\partial^2\xi}{\partial r\partial t}+\frac{d\nu}{dr}\frac{\partial\xi}{\partial t}=0,\quad -\frac{\partial^2\xi}{\partial r\partial r}+\frac{1}{2}\frac{d\mu}{dr}\frac{\partial\xi}{\partial r}=0,-2\frac{\partial^2\xi}{\partial\theta\partial t}=0,\quad -2\frac{\partial^2\xi}{\partial\varphi\partial t}=0,$$

$$-2\frac{\partial^2\xi}{\partial\theta\partial r}+\frac{d\lambda}{dr}\frac{\partial\xi}{\partial\theta}=0,\quad -2\frac{\partial^2\xi}{\partial\varphi\partial r}+\frac{d\lambda}{dr}\frac{\partial\xi}{\partial\varphi}=0$$

$$2\cot\theta\frac{\partial\xi}{\partial\varphi}-2\frac{\partial^2\xi}{\partial\varphi\partial\theta}=0,\quad -\frac{\partial^2\xi}{\partial\theta\partial\theta}-\frac{1}{2}e^{\lambda(r)-\mu(r)}\frac{d\lambda}{dr}\frac{\partial\xi}{\partial r}=0, \tag{17}$$

$$-\frac{\partial^2\xi}{\partial\varphi\partial\varphi}-\frac{1}{2}e^{\lambda(r)-\mu(r)}\frac{d\lambda}{dr}\frac{\partial\xi}{\partial r}\sin^2\theta-\sin\theta\cos\theta\frac{\partial\xi}{\partial\theta}=0,$$

$$\frac{1}{2}e^{\nu(r)-\mu(r)}\frac{d\nu}{dr}\frac{\partial\xi}{\partial r}-\frac{\partial^2\xi}{\partial t^2}=0.$$

$$\frac{\partial^2\eta^1}{\partial s^2}=0,\quad 2\frac{\partial^2\eta^1}{\partial t\partial s}+e^{\nu(r)-\mu(r)}\frac{d\nu}{dr}\frac{\partial\eta^0}{\partial s}=0,\quad 2\frac{\partial^2\eta^1}{\partial r\partial s}-\frac{\partial^2\xi}{\partial s^2}+\frac{d\mu}{dr}\frac{\partial\eta^1}{\partial s}=0,$$

$$2\frac{\partial^2\eta^1}{\partial\theta\partial s}-\frac{\partial\eta^2}{\partial s}e^{\lambda(r)-\mu(r)}\frac{d\lambda}{dr}=0,\quad 2\frac{\partial^2\eta^1}{\partial\varphi\partial s}-\frac{\partial\eta^3}{\partial s}e^{\lambda(r)-\mu(r)}\frac{d\lambda}{dr}\sin^2\theta=0,$$

$$2\frac{\partial^2\eta^1}{\partial r\partial t}-\frac{d\nu}{dr}\frac{\partial\eta^1}{\partial t}-2\frac{\partial^2\xi}{\partial t\partial s}+e^{\nu(r)-\mu(r)}\frac{d\nu}{dr}\frac{\partial\eta^0}{\partial r}+\frac{d\mu}{dr}\frac{\partial\eta^1}{\partial t}=0,$$

$$2\frac{\partial^2\eta^1}{\partial\theta\partial t}-\frac{\partial\eta^2}{\partial t}e^{\lambda(r)-\mu(r)}\frac{d\lambda}{dr}+e^{\nu(r)-\mu(r)}\frac{d\nu}{dr}\frac{\partial\eta^0}{\partial\theta}=0,$$

$$2\frac{\partial^2\eta^1}{\partial\varphi\partial t}-\frac{\partial\eta^3}{\partial t}e^{\lambda(r)-\mu(r)}\frac{d\lambda}{dr}\sin^2\theta+e^{\nu(r)-\mu(r)}\frac{d\nu}{dr}\frac{\partial\eta^0}{\partial\varphi}=0,$$

$$2\frac{\partial^2\eta^1}{\partial\theta\partial r}-\frac{d\lambda}{dr}\frac{\partial\eta^1}{\partial\theta}-2\frac{\partial^2\xi}{\partial\theta\partial s}-\frac{\partial\eta^2}{\partial r}e^{\lambda(r)-\mu(r)}\frac{d\lambda}{dr}+\frac{d\mu}{dr}\frac{\partial\eta^1}{\partial\theta}=0,$$

$$2\frac{\partial^2\eta^1}{\partial\varphi\partial r}-\frac{d\lambda}{dr}\frac{\partial\eta^1}{\partial\varphi}-2\frac{\partial^2\xi}{\partial\varphi\partial s}-\frac{\partial\eta^3}{\partial r}e^{\lambda(r)-\mu(r)}\frac{d\lambda}{dr}\sin^2\theta+\frac{d\mu}{dr}\frac{\partial\eta^1}{\partial\varphi}=0,$$

$$2\frac{\partial^2\eta^1}{\partial\varphi\partial\theta}-2\cot\theta\frac{\partial\eta^1}{\partial\varphi}-\frac{\partial\eta^2}{\partial\varphi}e^{\lambda(r)-\mu(r)}\frac{d\lambda}{dr}-\frac{\partial\eta^3}{\partial\theta}e^{\lambda(r)-\mu(r)}\frac{d\lambda}{dr}\sin^2\theta=0,$$

$$\frac{\partial^2\eta^1}{\partial t^2}-\frac{1}{2}e^{\nu(r)-\mu(r)}\frac{d\nu}{dr}\frac{\partial\eta^1}{\partial r}+\frac{1}{2}e^{\nu(r)-\mu(r)}\eta^1+e^{\nu(r)-\mu(r)}\frac{d\nu}{dr}\frac{\partial\eta^0}{\partial t}=0,$$
$$\frac{\partial^2\eta^1}{\partial r\partial r}+\frac{1}{2}\frac{d\mu}{dr}\frac{\partial\eta^1}{\partial r}-2\frac{\partial^2\xi}{\partial r\partial s}+\frac{1}{2}\frac{d^2\mu}{dr^2}\eta^1=0,$$
$$\frac{\partial^2\eta^1}{\partial\theta\partial\theta}+\frac{1}{2}e^{\lambda(r)-\mu(r)}\frac{d\lambda}{dr}\frac{\partial\eta^1}{\partial r}-\frac{\partial\eta^2}{\partial\theta}e^{\lambda(r)-\mu(r)}\frac{d\lambda}{dr}$$
$$-\frac{1}{2}\eta^1e^{\lambda(r)-\mu(r)}\left[\left(\frac{d\lambda}{dr}\right)^2-\frac{d\lambda}{dr}\frac{d\mu}{dr}\right]=0,$$
$$\frac{\partial^2\eta^1}{\partial\varphi\partial\varphi}+\frac{1}{2}e^{\lambda(r)-\mu(r)}\sin^2\theta\frac{d\lambda}{dr}\frac{\partial\eta^1}{\partial r}+\sin\theta\cos\theta\frac{\partial\eta^1}{\partial\theta}$$
$$-\frac{1}{2}\eta^1\sin^2\theta e^{\lambda(r)-\mu(r)}\left[\left(\frac{d\lambda}{dr}\right)^2-\frac{d\lambda}{dr}\frac{d\mu}{dr}\right]$$
$$-\frac{1}{2}\eta^2e^{\lambda(r)-\mu(r)}\frac{d\lambda}{dr}2\sin\theta\cos\theta-\frac{\partial\eta^3}{\partial\varphi}e^{\lambda(r)-\mu(r)}\frac{d\lambda}{dr}\sin^2\theta=0,$$
$$-\frac{1}{2}e^{\lambda(r)-\mu(r)}\frac{\partial\xi}{\partial r}\frac{d\lambda}{dr}-\frac{\partial^2\xi}{\partial\theta\partial\theta}=0,\quad -2\frac{\partial^2\xi}{\partial r\partial t}+\frac{d\nu}{dr}\frac{\partial\xi}{\partial t}=0,\quad -2\frac{\partial^2\xi}{\partial\theta\partial t}=0,$$
$$-\frac{\partial^2\xi}{\partial\varphi\partial\varphi}-\frac{1}{2}\sin^2\theta e^{\lambda(r)-\mu(r)}\frac{d\lambda}{dr}\frac{\partial\xi}{\partial r}-\sin\theta\cos\theta\frac{\partial\xi}{\partial\theta}=0,\quad -2\frac{\partial^2\xi}{\partial\varphi\partial t}=0,\tag{18}$$
$$\frac{1}{2}e^{\nu(r)-\mu(r)}\frac{d\nu}{dr}\frac{\partial\xi}{\partial r}-\frac{\partial^2\xi}{\partial t^2}=0,\quad 2\cot\theta\frac{\partial\xi}{\partial\varphi}-2\frac{\partial^2\xi}{\partial\varphi\partial\theta}=0,$$
$$\frac{d\lambda}{dr}\frac{\partial\xi}{\partial\theta}-2\frac{\partial^2\xi}{\partial\theta\partial r}=0,\quad \frac{d\lambda}{dr}\frac{\partial\xi}{\partial\varphi}-2\frac{\partial^2\xi}{\partial\varphi\partial r}=0,\quad \frac{1}{2}\frac{d\mu}{dr}\frac{\partial\xi}{\partial r}-\frac{\partial^2\xi}{\partial r\partial r}=0.$$

$$\frac{\partial^2\eta^2}{\partial s^2}=0,\quad 2\frac{\partial^2\eta^2}{\partial t\partial s}=0,\quad 2\frac{\partial^2\eta^2}{\partial r\partial s}+\frac{\partial\eta^2}{\partial s}\frac{d\lambda}{dr}=0,$$
$$2\frac{\partial^2\eta^2}{\partial\theta\partial s}-\frac{\partial^2\xi}{\partial s^2}+\frac{\partial\eta^1}{\partial s}\frac{d\lambda}{dr}=0,$$
$$2\frac{\partial^2\eta^2}{\partial\varphi\partial s}-2\sin\theta\cos\theta\frac{\partial\eta^3}{\partial s}=0,\quad 2\frac{\partial^2\eta^2}{\partial\varphi\partial t}-2\sin\theta\cos\theta\frac{\partial\eta^3}{\partial t}=0,$$
$$\eta^1\frac{d^2\lambda}{dr^2}+\frac{\partial\eta^1}{\partial r}\frac{d\lambda}{dr}+2\frac{\partial^2\eta^2}{\partial\theta\partial r}-2\frac{\partial^2\xi}{\partial r\partial s}=0,$$
$$\frac{\partial\eta^1}{\partial\varphi}\frac{d\lambda}{dr}-2\sin\theta\cos\theta\frac{\partial\eta^3}{\partial\theta}+2\frac{\partial^2\eta^2}{\partial\varphi\partial\theta}-2\cot\theta\frac{\partial\eta^2}{\partial\varphi}-2\frac{\partial^2\xi}{\partial\varphi\partial s}=0,$$
$$-2\sin\theta\cos\theta\frac{\partial\eta^3}{\partial r}+2\frac{\partial^2\eta^2}{\partial\varphi\partial r}=0,\quad 2\frac{\partial^2\eta^2}{\partial r\partial t}-\frac{d\nu}{dr}\frac{\partial\eta^2}{\partial t}+\frac{\partial\eta^2}{\partial t}\frac{d\lambda}{dr}=0,$$
$$2\frac{\partial^2\eta^2}{\partial\theta\partial t}-2\frac{\partial^2\xi}{\partial t\partial s}+\frac{\partial\eta^1}{\partial t}\frac{d\lambda}{dr}=0,\quad \frac{\partial^2\eta^2}{\partial t^2}-\frac{1}{2}e^{\nu(r)-\mu(r)}\frac{d\nu}{dr}\frac{\partial\eta^2}{\partial r}=0,$$
$$\frac{\partial^2\eta^2}{\partial r\partial r}-\frac{1}{2}\frac{d\mu}{dr}\frac{\partial\eta^2}{\partial r}+\frac{\partial\eta^2}{\partial r}\frac{d\lambda}{dr}=0,$$
$$\frac{\partial^2\eta^2}{\partial\theta\partial\theta}+\frac{1}{2}e^{\lambda(r)-\mu(r)}\frac{d\lambda}{dr}\frac{\partial\eta^2}{\partial r}-2\frac{\partial^2\xi}{\partial\theta\partial s}+\frac{\partial\eta^1}{\partial\theta}\frac{d\lambda}{dr}=0,$$
$$\frac{\partial^2\eta^2}{\partial\varphi\partial\varphi}+\frac{1}{2}\sin^2\theta e^{\lambda(r)-\mu(r)}\frac{d\lambda}{dr}\frac{\partial\eta^2}{\partial r}+\sin\theta\cos\theta\frac{\partial\eta^2}{\partial\theta}-\eta^2\cos2\theta$$

$$-2\sin\theta\cos\theta\frac{\partial\eta^3}{\partial\varphi}=0,\quad -2\frac{\partial^2\xi}{\partial r\partial t}+\frac{d\nu}{dr}\frac{\partial\xi}{\partial t}=0,\quad -2\frac{\partial^2\xi}{\partial\varphi\partial t}=0,$$

$$\frac{1}{2}\frac{d\mu}{dr}\frac{\partial\xi}{\partial r}-\frac{\partial^2\xi}{\partial r\partial r}=0,\quad -2\frac{\partial^2\xi}{\partial\theta\partial r}+\frac{d\lambda}{dr}\frac{\partial\xi}{\partial\theta}=0,\quad \frac{d\lambda}{dr}\frac{\partial\xi}{\partial\varphi}-2\frac{\partial^2\xi}{\partial\varphi\partial r}=0,$$

$$2\cot\theta\frac{\partial\xi}{\partial\varphi}-2\frac{\partial^2\xi}{\partial\varphi\partial\theta}=0,\quad \frac{1}{2}e^{\nu(r)-\mu(r)}\frac{\partial\xi}{\partial r}\frac{d\nu}{dr}-\frac{\partial^2\xi}{\partial t^2}=0,$$

$$-\frac{\partial^2\xi}{\partial\varphi\partial\varphi}-\frac{1}{2}e^{\lambda(r)-\mu(r)}\frac{d\lambda}{dr}\frac{\partial\xi}{\partial r}\sin^2\theta-\sin\theta\cos\theta\frac{\partial\xi}{\partial\theta}=0,$$

$$-\frac{1}{2}e^{\lambda(r)-\mu(r)}\frac{d\lambda}{dr}\frac{\partial\xi}{\partial r}-\frac{\partial^2\xi}{\partial\theta\partial\theta}=0. \tag{19}$$

$$\frac{\partial^2\eta^3}{\partial s^2}=0,\quad 2\frac{\partial^2\eta^3}{\partial t\partial s}=0,\quad 2\frac{\partial^2\eta^3}{\partial r\partial s}+\frac{\partial\eta^3}{\partial s}\frac{d\lambda}{dr}=0,\quad 2\frac{\partial^2\eta^3}{\partial\theta\partial s}+2\cot\theta\frac{\partial\eta^3}{\partial s}=0,$$

$$\frac{\partial\eta^1}{\partial s}\frac{d\lambda}{dr}+2\cot\theta\frac{\partial\eta^2}{\partial s}+2\frac{\partial^2\eta^3}{\partial\varphi\partial s}-\frac{\partial^2\xi}{\partial s^2}=0,$$

$$\frac{\partial\eta^1}{\partial\theta}\frac{d\lambda}{dr}-2\eta^2\frac{1}{\sin^2\theta}+2\cot\theta\frac{\partial\eta^2}{\partial\theta}+2\frac{\partial^2\eta^3}{\partial\varphi\partial\theta}-2\frac{\partial^2\xi}{\partial\theta\partial s}=0,$$

$$\eta^1\frac{d^2\lambda}{dr^2}+\frac{\partial\eta^1}{\partial r}\frac{d\lambda}{dr}+2\cot\theta\frac{\partial\eta^2}{\partial r}+2\frac{\partial^2\eta^3}{\partial\varphi\partial r}-2\frac{\partial^2\xi}{\partial r\partial s}=0,$$

$$2\cot\theta\frac{\partial\eta^3}{\partial r}+2\frac{\partial^2\eta^3}{\partial\theta\partial r}=0,\quad 2\frac{\partial^2\eta^3}{\partial r\partial t}-\frac{\partial\eta^3}{\partial t}\frac{d\nu}{dr}+\frac{\partial\eta^3}{\partial t}\frac{d\lambda}{dr}=0,$$

$$2\cot\theta\frac{\partial\eta^3}{\partial t}+2\frac{\partial^2\eta^3}{\partial\theta\partial t}=0,\quad \frac{\partial\eta^1}{\partial t}\frac{d\lambda}{dr}+2\cot\theta\frac{\partial\eta^2}{\partial t}-2\frac{\partial^2\xi}{\partial t\partial s}+2\frac{\partial^2\eta^3}{\partial\varphi\partial t}=0,$$

$$\frac{\partial^2\eta^3}{\partial t^2}-\frac{1}{2}e^{\nu(r)-\mu(r)}\frac{\partial\eta^3}{\partial r}\frac{d\nu}{dr}=0,\quad \frac{\partial^2\eta^3}{\partial r\partial r}-\frac{1}{2}\frac{\partial\eta^3}{\partial r}\frac{d\mu}{dr}+\frac{\partial\eta^3}{\partial r}\frac{d\lambda}{dr}=0,$$

$$\frac{1}{2}e^{\lambda(r)-\mu(r)}\frac{\partial\eta^3}{\partial r}\frac{d\lambda}{dr}+\frac{\partial^2\eta^3}{\partial\theta\partial\theta}+2\cot\theta\frac{\partial\eta^3}{\partial\theta}=0,$$

$$\frac{1}{2}\sin^2\theta e^{\lambda(r)-\mu(r)}\frac{\partial\eta^3}{\partial r}\frac{d\lambda}{dr}+\frac{\partial^2\eta^3}{\partial\varphi\partial\varphi}+\sin\theta\cos\theta\frac{\partial\eta^3}{\partial\theta}+2\cot\theta\frac{\partial\eta^2}{\partial\varphi}$$

$$+\frac{\partial\eta^1}{\partial\varphi}\frac{d\lambda}{dr}-2\frac{\partial^2\xi}{\partial\varphi\partial s}=0,$$

$$-2\frac{\partial^2\xi}{\partial r\partial t}+\frac{d\nu}{dr}\frac{\partial\xi}{\partial t}=0,-\frac{\partial^2\xi}{\partial r\partial r}+\frac{1}{2}\frac{d\mu}{dr}\frac{\partial\xi}{\partial r}=0$$

$$-2\frac{\partial^2\xi}{\partial\varphi\partial r}+\frac{d\lambda}{dr}\frac{\partial\xi}{\partial\varphi}=0,\quad -2\frac{\partial^2\xi}{\partial\varphi\partial t}=0,\frac{1}{2}e^{\nu(r)-\mu(r)}\frac{d\nu}{dr}\frac{\partial\xi}{\partial r}-\frac{\partial^2\xi}{\partial t^2}=0,$$

$$\frac{\partial^2\xi}{\partial\theta\partial t}=0,\quad 2\frac{\partial^2\xi}{\partial\theta\partial r}+\frac{d\lambda}{dr}\frac{\partial\xi}{\partial\theta}=0,$$

$$-\frac{\partial^2\xi}{\partial\theta\partial\theta}-\frac{1}{2}e^{\lambda(r)-\mu(r)}\frac{d\lambda}{dr}\frac{\partial\xi}{\partial r}=0,\quad -2\frac{\partial^2\xi}{\partial\varphi\partial\theta}+2\cot\theta\frac{\partial\xi}{\partial\varphi}=0,$$

$$-\frac{\partial^2\xi}{\partial\varphi\partial\varphi}-\sin\theta\cos\theta\frac{\partial\xi}{\partial\theta}-\frac{1}{2}e^{\lambda(r)-\mu(r)}\frac{d\lambda}{dr}\frac{\partial\xi}{\partial r}\sin^2\theta=0. \tag{20}$$

We intend to classify of the four-dimensional static space-times in spherical coordinate system with respect to their Lie symmetries by finding the solutions of on system of PDEs.

The Lie symmetry determination equations (17) - (20) for the four-dimensional static spacetime are both very complex systems of partial differential equations. Finding the solution to the problem is generally very complex and requires the use of computational software. Equations (17) ~ (20) have the following solutions:

$$\xi = 1, \eta^0 = \eta^1 = \eta^2 = \eta^3 = 0, \quad \mathbf{X}_0 = \frac{\partial}{\partial s}; \tag{21}$$

$$\eta^0 = 1, \xi = \eta^1 = \eta^2 = \eta^3 = 0, \quad \mathbf{X}_1 = \frac{\partial}{\partial t}; \tag{22}$$

$$\eta^3 = 1, \xi = \eta^0 = \eta^1 = \eta^2 = 0, \quad \mathbf{X}_2 = \frac{\partial}{\partial \varphi}; \tag{23}$$

$$\eta^2 = \cos\varphi, \eta^3 = -\cot\theta\sin\varphi, \xi = \eta^0 = \eta^1 = 0, \mathbf{X}_3 = -\cos\varphi\frac{\partial}{\partial\theta} + \cot\theta\sin\varphi\frac{\partial}{\partial\varphi}; \tag{24}$$

$$\eta^2 = \sin\varphi, \eta^3 = \cot\theta\cos\varphi, \xi = \eta^0 = \eta^1 = 0, \mathbf{X}_4 = \sin\varphi\frac{\partial}{\partial\theta} + \cot\theta\cos\varphi\frac{\partial}{\partial\varphi}. \tag{25}$$

The Lie symmetry generator operator of the four-dimensional static space-time in spherical coordinate system forms a five-dimensional Lie algebra, which is written as:

$$[\mathbf{X}_i,\mathbf{X}_0] = 0, \quad [\mathbf{X}_1,\mathbf{X}_0] = 0, \quad [\mathbf{X}_2,\mathbf{X}_3] = -\mathbf{X}_4, \quad [\mathbf{X}_2,\mathbf{X}_4] = -\mathbf{X}_3, \quad [\mathbf{X}_3,\mathbf{X}_4] = -\mathbf{X} \quad i=1,2,3 \tag{26}$$

The Eqs.(21)-(25) are minimal set, which is 5, of Lie symmetries for the general four-dimensional static spacetime in spherical coordinate system.

The Eqs. (21)~(25) can give the required classification of the four-dimensional static spacetime in spherical coordinate system according to the Lie symmetries.

Lie Theorem of the Four-Dimensional Static Spacetime in Spherical Coordinate System

Theorem: For the generators $\xi, \eta^0, \eta^1, \eta^2, \eta^3$ of infinitesimal transformation of four-dimensional static spacetime in spherical coordinate system which satisfy the determining equations (13)-(16), if there is gauge function $G(s,t,r,\theta,\varphi)$ satisfies the following equation:

$$L\dot{\xi} + X^{(1)}(L) + \sum_{k=0}^{3} Q_k\left(\dot{\eta}^k - \dot{q}_k\dot{\xi}\right) + \dot{G} = 0, \tag{27}$$

then the static spacetime system in spherical coordinate system possesses the following conserved quantity:

$$I = L\xi + \sum_{k=0}^{3}\frac{\partial L}{\partial \dot{q}_k}\left(\eta^k - \dot{q}_k\xi\right) + G = \text{const}. \tag{28}$$

Proof:

$$\frac{\mathrm{d}I}{\mathrm{d}s} = \dot{L}\xi + L\dot{\xi} + \sum_{k=0}^{3}\frac{\mathrm{d}}{\mathrm{d}s}\frac{\partial L}{\partial \dot{q}_k}\left(\eta^k - \dot{q}_k\xi\right) + \sum_{k=0}^{3}\frac{\partial L}{\partial \dot{q}_k}\left(\dot{\eta}^k - \ddot{q}_k\xi - \dot{q}_k\dot{\xi}\right) + \dot{G}, \tag{29}$$

Substitution of Eq.(27) to Eq.(29) and making further simplification, we obtain,

$$\frac{\mathrm{d}I}{\mathrm{d}s} = \sum_{k=1}^{4}\left(\xi^k - \dot{q}_k\xi_0\right)\left(\frac{\mathrm{d}L}{\mathrm{d}s}\frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} - Q_K\right) = 0.$$

Equation (27) is called the structure equation of Lie symmetries of four-dimensional static spacetime in spherical coordinate system.

From Eqs. (28), it can be seen that there are four conserved quantities containing metric coefficients $\nu(r)$, $\mu(r)$, and $\lambda(r)$ in four-dimensional static spacetime for spherical coordinates. The metric coefficients can take any value, which means there are countless conserved quantities in four-dimensional static spacetime for spherical coordinates. If the metric coefficients are assigned specific values,the four-dimensional static spacetime in spherical coordinates can be classified. Since the classification of four-dimensional static spacetime has already been studied using Noether symmetry method, this article will not discuss this classification problem. On the other hand, when we determine the metric coefficients in equation (28), the five conserved variables are all very simple first-order differential equations. With the given initial conditions, we can easily obtain the exact solution of four-dimensional static spatiotemporal space in spherical coordinate system.

Conclusion

This article uses the method of analytical mechanics to study the gravitational field problem in four-dimensional space-time, and provides several important results: firstly, by introducing four-dimensional space-time coordinates as generalized coordinates and taking curve coordinates as independent variables, the Lagrange equations (5)-(8) of the static space-time in a spherical coordinate system are established; the secondly is to introduce transformation of the Lie groups and corresponding vector fields related to curve coordinates and generalized coordinates for four-dimensional space-time; thirdly, based on the invariance of the Lagrange equations of the static space-time in a spherical coordinate system under the transformation of the Lie group, the Lie symmetry determination equations (13)-(16) and a series of symmetry killing equations (17)-(20) for the four-dimensional static space-time in a spherical coordinate system are given; fourthly proposed and proved the Lie symmetry theorem for the plane symmetry static space-time in a spherical coordinate system, and provided the structural equation (27) for the existence of the gravitational fields and the form of conserved quantities (28) [37-46].

References

1. Kramer, D., Stephani, H., MacCallum, M., & Herlt, E. (1980). Exact solutions of Einstein's field equations. Berlin.
2. Mahmood, A., Siddiqui, A. A., & Feroze, T. (2017). Non-static spherically symmetric exact solution of the Einstein-Maxwell field equations. Journal of the Korean Physical Society, 71(7), 396-403.
3. Feroze, T., Mahomed, F. M., & Qadir, A. (2006). The connection between isometries and symmetries of geodesic equations of the underlying spaces. Nonlinear Dynamics, 45(1), 65-74.
4. Qadir, A., & Ziad, M. (1995). The classification of spherically symmetric space-times. Il Nuovo Cimento B (1971-1996), 110(3), 317-334.
5. Qadir, A., Saifullah, K., & Ziad, M. (2003). Classification of cylindrically symmetric static spacetimes according to their Ricci collineations. General Relativity and Gravitation, 35(11), 1927-1975.
6. Foyster, J. M., & McIntosh, C. B. G. (1973). The classification of some spherically symmetric space-time metrics. Bulletin of the Australian Mathematical Society, 8(2), 187-190.
7. Bokhari, A. H., Kashif, A. R., & Qadir, A. (2000). Classification of curvature collineations of plane symmetric static spacetimes. Journal of Mathematical Physics, 41(4), 2167-2172.
8. Lie, S., & Engel, F. (1893). Theorie der transformationsgruppen (Vol. 3). Teubner.
9. Ovisiannikov, L. V. (1982). Group analysis of difference equations. New York: Academic.
10. Ibragimov, N.H. CRC Handbook of Lie group analysis of differential equations, Volume 1:symmetries, exact solutions and conservation laws; CRC Press: Boca Raton, FL, USA, 1994.
11. Ibragimov, N. H., Kovalev, V. F., & Pustovalov, V. V. (2002). Symmetries of integro-differential equations: A survey of methods illustrated by the Benny equations. Nonlinear Dynamics, 28(2), 135-153.
12. Olver P.J., Applications of Lie group to differential equations, New York, Springer-Verlag, 1986
13. Bluman G.W., Kumei S., Symmetries and differential equations, New York, Springer-Verlag, 1989
14. Stephani, H., Differential equations their solution using symmetries, Cambridge University Press, 1990.
15. Ibragimov, N. K., & Ibragimov, N. K. (1999). Elementary Lie group analysis and ordinary differential equations (Vol. 197). New York: Wiley.
16. Noether, E. (1971). Invariant variation problems. Transport theory and statistical physics, 1(3), 186-207.
17. Mei, F. X. (1999). Applications of Lie groups and Lie algebras to constrained mechanical systems.
18. Lutzky M., Dynamical symmetries and conserved quantities, J. Phys., A: Math. Gen., 1979,12(7):973-981
19. Djukic, D. S., & Vujanovic, B. D. (1975). Noether's theory in classical nonconservative mechanics. Acta Mechanica, 23(1), 17-27.
20. Fu, J. L., & Chen, L. Q. (2003). Non-Noether symmetries and conserved quantities of nonconservative dynamical systems. Physics Letters A, 317(3-4), 255-259.
21. Yong-Xin, G., Li-Yan, J., & Ying, Y. (2001). Symmetries of mechanical systems with nonlinear nonholonomicconstraints. Chinese Physics, 10(3), 181.
22. Rong-Wan, L., & Li-Qun, C. (2004). Lie symmetries and invariants of constrained Hamiltonian systems. Chinese Physics, 13(10), 1615.
23. Xiang-Wei, C., & Yan-Min, L. (2003). Perturbation to symmetries and adiabatic invariantsof a type of nonholonomic singular system. Chinese Physics, 12(12), 1349.
24. Zhou, S., Fu, H., & Fu, J. (2011). Symmetry theories of Hamiltonian systems with fractional derivatives. Science China Physics, Mechanics and Astronomy, 54(10), 1847.

25. Fu, J. L., Liu, R. W., & Mei, F. X. (1998). Lie symmetries and conserved quantities of holonomic mechanical system in terms of quasi-coordinates. J of BIT, 7(3), 215-220.
26. Fu, J., Jiménez, S., Tang, Y., & Vázquez, L. (2008). Construction of exact invariants of time-dependent linear nonholonomic dynamical systems. Physics Letters A, 372(10), 1555-1561.
27. Fu, J. L., Fu, L. P., Chen, B. Y., & Sun, Y. (2016). Lie symmetries and their inverse problems of nonholonomic Hamilton systems with fractional derivatives. Physics Letters A, 380(1-2), 15-21.
28. Fu J. L., Zhang L.J., Cao S., Xiang C. and Zao W.J.. A symplectic algorithm for constrained Hamiltonian systems, Axioms,2022,11,217
29. Cao, S., & Fu, J. L. (2018). Symmetry theories for canonicalized equations of constrained Hamiltonian system. Nonlinear Dynamics, 92(4), 1947-1954.
30. Cai, P. P., Fu, J. L., & Guo, Y. X. (2017). Lie symmetries and conserved quantities of the constraint mechanical systems on time scales. Reports on Mathematical Physics, 79(3), 279-298.
31. Fu, J. L., Zhang, L., Khalique, C. M., & Guo, M. L. (2021). Motion equations and non-Noether symmetries of Lagrangian systems with conformable fractional derivative. Thermal Science, 25(2 Part B), 1365-1372.
32. Zhou, S., Fu, H., & Fu, J. (2011). Symmetry theories of Hamiltonian systems with fractional derivatives. Science China Physics, Mechanics and Astronomy, 54(10), 1847.
33. Fu, J. L., Fu, H., & Liu, R. W. (2010). Hojman conserved quantities of discrete mechanico–electrical systems constructed by continuous symmetries. Physics Letters A, 374(17-18), 1812-1818.
34. Jing-Li, F., Gui-Dong, D., Jimenez, S., & Yi-Fa, T. (2007). Discrete variational principle and first integralsfor Lagrange–Maxwell mechanico-electrical systems. Chinese Physics, 16(3), 570.
35. Fu, J., Chen, B., Fu, H., Zhao, G., Liu, R., & Zhu, Z. (2011). Velocity-dependent symmetries and non-Noether conserved quantities of electromechanical systems. Science China Physics, Mechanics and Astronomy, 54(2), 288-295.
36. Fu, J. L., Lu, X. D., Xiang, C., & Guo, Y. X. (2022). Lie group analysis method for wall climbing robot systems. Indian Journal of Physics, 96(14), 4231-4243.
37. Jing-Li, F., Chun, X., & Lei, M. (2021). Algebraic structure and Poisson integral method of snake-like robot systems. Frontiers in Physics, 9, 643016.
38. Bokhari, A. H., & Kashif, A. R. (1996). Curvature collineations of some static spherically symmetric space–times. Journal of Mathematical Physics, 37(7), 3498-3504.
39. Feroze, T., Qadir, A., & Ziad, M. (2001). The classification of plane symmetric spacetimes by isometries. Journal of Mathematical Physics, 42(10), 4947-4955.
40. Qadir, A., & Ziad, M. (1995). Classification of static cylindrically symmetric space-times. Il Nuovo Cimento B (1971-1996), 110(3), 277-290.
41. Goyal, N., & Gupta, R. K. (2012). New exact solutions of the Einstein—Maxwell equations for magnetostatic fields. Chinese Physics B, 21(9), 090401.
42. Capozziello, S., Frusciante, N., & Vernieri, D. (2012). New spherically symmetric solutions in f (R)-gravity by Noether symmetries. General Relativity and Gravitation, 44(8), 1881-1891.
43. Ali, F., & Feroze, T. (2013). Classification of plane symmetric static space-times according to their Noether symmetries. International Journal of Theoretical Physics, 52(9), 3329-3342.
44. Feroze, T., & Ali, F. (2014). Corrigendum to "Noether symmetries and conserved quantities for spaces with a section of zero curvature"[J. Geom. Phys. 61 (2011) 658–662]. Journal of Geometry and Physics, 80, 88-89.
45. Ali, F., & Feroze, T. (2016). Complete classification of cylindrically symmetric static spacetimes and the corresponding conservation laws. Mathematics, 4(3), 50.
46. Bokhari, A. H., Kara, A. H., Kashif, A. R., & Zaman, F. D. (2007). On the symmetry structures of the Minkowski metric and a Weyl re-scaled metric. International Journal of Theoretical Physics, 46(11), 2795-2800.

Foot Notes

1. This work was supported by the Natural Science Foundation of China (11872335, 11972177) and Key Projects of the National Natural Science Foundation of China (11232009), and supported by the Natural Science Foundation of Shandong Province of China (ZR2023MA070)

2. Statement of Conflict of Interest: The Authors declare no conflict of interest.

Data Availability Statement: The data is available from the authors on request.

Authors Contribution: All authors contributed equally in this article.