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Paracorner Manifolds

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Abstract
The global structure of a 3-dimensional paracorner manifold is described using a special adapted global frame. This enables a description of the curvature of such manifolds. We explore the presence of generalized Ricci-Yamabe solitons (briefly, GRYS) on such a manifold in dimension three. Concrete examples are given.

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Introduction
C_{12}-manifolds or corner manifolds are odd dimensional differentiable Riemannian manifolds which are integrable non normal almost contact metric manifolds. This concept was first introduced in the classification of ChineaGonzalez [1]. The exploration of geometric properties on these manifolds constitutes a broad and dynamic area of research, drawing considerable attention in recent literature such as [2-6].

On the opposite side, that is, in the paracontact manifolds, and as is the case with the paraSasakian, paraKenmotsu, and paracosymplectic manifolds, we can establish what corresponds to C12-manifolds, and we will call it paracorner manifolds. Here we will present a comprehensive study of such manifolds with concrete examples [7-9].

For the second part of this paper, and considering that the generalized Ricci soliton equation (briefly, GRS), as formulated in a pseudo-Riemannian manifold (M^n,g), is characterized by its definition (see)

$$\mathcal{L}_V g = -2c_1 V^\flat \otimes V^\flat + 2c_2 S + 2\lambda g, \tag{1.1}$$

where $L_V\, g$ is the Lie derivative of the metric g along the vector field V

$$(\mathcal{L}_V g)(X,Y) = g\big(\nabla_X V, Y\big) + g\big(X, \nabla_Y V\big), \tag{1.2}$$

and $V^\flat$ is the g-dual of the vector V

$V^\flat(X) = g(X, V)$ [10].

Equation (1.1) is a generalization of:
- Killing's equation $c_1 = c_2 = \lambda = 0$.
- Equation for homotheties $c_1 = c_2 = 0$.
- Ricci soliton $c_1 = 0$, $c_2 = -1$.
- Cases of Einstein-Weyl $c_1 = 1,\ c_2 = \frac{-1}{n-2}$.
- Metric projective structures with skew-symmetric Ricci tensor in projective class $c_1 = 1,\ c_2 = -\frac{1}{n-1},\ \lambda = 0$.

- Vacuum near-horizon geometry equation $c_1 = 1,\ c_2 = \frac{1}{2}$

Also, a Ricci-Yamabe soliton (briefly, RYS) is defined as a semi-Riemannian manifold (Mn,g) equipped with a vector field V on M that satisfies

$$\mathcal{L}_V g = 2\alpha S + 2(\lambda + \mu r)g, \tag{1.3}$$

where $\mu \in$ R is constant and r denotes the scalar curvature, defined as the trace of the Ricci tensor S with respect to the metric g

$$r = Tr_g S. \tag{1.4}$$

Likewise, equation (1.3) is a natural generalization of:
- Ricci Soliton (briefly, RS) $\alpha = 1, \mu = 0$.
- Ricci-Bourguignon Soliton (briefly, GBS) $\alpha = 1, \mu \in R$.
- Yamabe Soliton (briefly, YS) $\alpha = 0, \mu = -1$.

Motivated by the work of, we define the concept of a generalized Ricci-Yamabe soliton (briefly, GRYS) as follows [2,11]:

$$\mathcal{L}_V g = -2c_1 V^\flat \otimes V^\flat + 2c_2 S + 2(\lambda + \mu r)g, \tag{1.5}$$

where c_1, c_2, λ, μ are constants. It is easy to see that all of the above cases are special cases of (1.5), especially (1.1) and (1.3). In the last section, we will provide proof of the existence of this generalization.

First, we will introduce the basic concepts that we need in this research.
- Preliminaries
- Paracorner manifolds

An odd-dimensional Riemannian manifold (M^{2n+1}, g) is said to be an almost paracontact metric manifold if there exist on M a (1,1)-tensor field ϕ, a vector field ξ (called the structure vector field) and a 1-form η such that

$$\begin{cases} \eta(\xi) = 1, \\ \varphi^2(X) = X - \eta(X)\xi, \\ g(\varphi X, \varphi Y) = -g(X,Y) + \eta(X)\eta(Y), \end{cases} \tag{3.1}$$

for any vector fields X, Y on M. In particular, in an almost paracontact metric manifold we also have

$$\varphi\xi = 0 \qquad and \qquad \eta \circ \varphi = 0 \tag{3.2}$$

The fundamental 2-form φ is defined by

$$\phi(X,Y) = g(\varphi X, Y). \tag{3.3}$$

As shown in, any almost paracontact metric manifold $(M^{2n+1}, \varphi, \xi, \eta, g)$ admits a φ-basis, that is, a local orthonormal basis of the form $\{\xi, e_i, \varphi e_i\}_{1 \le i \le 2n}$ where ξ and ei are space-like vector fields and φe_i are time-like vector fields [12].

It is known that the almost paracontact structure (ϕ, ξ, η) is said to be normal if and only if

$$N^{(1)}(X,Y) = N_\varphi(X,Y) - 2d\eta(X,Y)\xi = 0, \tag{3.4}$$

for any X, Y on M, where N_ϕ denotes the Nijenhuis torsion of ϕ, given by

$$N_\varphi(X,Y) = \varphi^2[X,Y] + [\varphi X, \varphi Y] - \varphi\big([\varphi X, Y] + [X, \varphi Y]\big). \tag{3.5}$$

Just like almost contact metric structures, there is a fundamental Lemma related to almost paracontact structures which we present as follows:

Lemma 3.1. For every almost paracontact metric manifold (M,ϕ,ξ,η,g), we have

$$\begin{aligned} 2g\big((\nabla_X \varphi)Y, Z\big) &= -3\mathrm{d}\phi(X, \varphi Y, \varphi Z) - 3\mathrm{d}\phi(X, Y, Z) \\ &\quad -g\big(N^{(1)}(Y,Z), \varphi X\big) + N^{(2)}(Y,Z)\eta(X) \end{aligned}$$

$$+2\mathrm{d}\eta(\varphi Y, X)\eta(Z) - 2\mathrm{d}\eta(\varphi Z, X)\eta(Y), \qquad (3.6)$$

for all X,Y and Z vector fields on M.

Proof. Proof is similar to the proof of analogous formula in case of almost contact metric manifolds. The latter can be found in (, p. 82) [8].

Some special cases are worthy of attention. A paracontact metric structure is an almost paracontact metric structure with dη = ϕ and a ParaSasakian manifold, is a normal paracontact metric manifold, equivalently, if

$$(\nabla_X \varphi)Y = -g(X,Y)\xi + \eta(Y)X.$$

An almost paracontact metric structure is said to be almost paracosymplectic, if both η and φ are closed. If in addition the structure is normal, the structure is said to be paracosymplectic, equivalently, $(\nabla_x \varphi)Y = 0$ [9].

An almost paracontact metric manifold is an almost paraKenmotsu manifold, if dη = 0 and dφ = 2η ∧ φ [8].

Moreover, If the structure (φ,ξ,η) is normal, the manifold is said to be Kenmotsu. This manifold, can be characterized through their Levi-Civita connection, by requiring

$$(\nabla_X \varphi)Y = g(X, \varphi Y)\xi + \eta(Y)\varphi X.$$

The above structures paraSasakian, paraKenmotsu and paracosymplectic are in the classes F_4, F_5 and F_0 respectively, of the classification of G. Nakova [13]. In this classification, the almost paracontact metric structures have been completely classified. Most of the research related to almost paracontact metric structures is concerned with the normal structures. In this paper, we focus on the class F_{11} which is integrable but not normal. This class is characterized by:

$$(\nabla_X \phi)(Y,Z) = \eta(X)\big(\eta(Y)\tilde{\omega}(Z) - \eta(Z)\tilde{\omega}(Y)\big), \qquad (3.7)$$

for all X,Y and Z vector field on M and $\tilde{\omega}(X) = (\nabla_\xi \phi)(\xi, X)$. Then, with the use of (3.1), we get

$$\begin{aligned} \tilde{\omega}(X) &= g(\nabla_\xi \varphi)\xi, X) \\ &= g(\nabla_\xi \xi, \varphi X). \end{aligned}$$

It is easy to see that $g(\nabla_\xi \xi, \xi) = 0$ i.e., ξ is orthogonal to $\nabla_\xi \xi$. For the sake of simplicity, we will denote by ψ the vector field $\nabla_\xi \xi$, which represents the mean curvature vector field of any integral curve of $D^\perp$ where D and $D^\perp$ denotes the mutually orthogonal distributions associated with the subbundles Kerη and span{ξ} of the tangent bundle TM, respectively.

On the other hand, the existence of three global vector fields ξ, ψ and ϕψ which are naturally pairwise orthogonal, encourages us to suggest the name ”paracorner manifold” instead of F11-manifold. Based on these facts, one can get the following:

Theorem 3.2. The manifold M endowed with an almost paracontact metric structure (φ,ξ,η,g) will be called paracorner manifold if and only if

$$(\nabla_X \varphi)Y = -\eta(X)\big(\eta(Y)\varphi\psi + \omega(\varphi Y)\xi\big), \qquad (3.8)$$

for all X,Y and Z vector fields on M and $\omega(X) = g(\psi, X) = -\tilde{\omega}(\varphi X)$.

Proof. The proof is direct, using (3.3) we get

$$\begin{aligned} (\nabla_X \phi)(Y,Z) &= X\phi(Y,Z) - \phi(\nabla_X Y, Z) - \phi(Y, \nabla_X Z) \\ &= g\big((\nabla_X \varphi)Y, Z\big). \end{aligned}$$

Now, just use (3.7).
Corollary 3.3. For any paracorner manifold we have

$$\nabla_X \xi = \eta(X)\psi \qquad and \qquad \nabla_{\varphi X}\xi = 0. \qquad (3.9)$$

Proof. Assuming (M,φ,ξ,η,g) is a paracorner manifold. Then, we have

$$(\nabla_X \varphi)Y \quad = \quad -\eta(X)\big(\eta(Y)\varphi\psi + \omega(\varphi Y)\xi\big),$$

setting $Y = \xi$ gives

$$\varphi\nabla_X\xi = \eta(X)\varphi\psi,$$

and hence

$$\begin{aligned}\nabla_X\xi &= \eta(X)\varphi^2\psi \\ &= \eta(X)\psi. \end{aligned} \tag{3.10}$$

To obtain the second formula, replace X by φX in the first one.
Now, we give a new characterization of paracorner manifolds.

Theorem 3.4. An almost paracontact metric manifold is a paracorner manifold if and only if there exists a 1-form ω such that

$$\mathrm{d}\eta = \eta\wedge\omega \qquad \mathrm{d}\phi = 0 \qquad and \qquad N_\varphi = 0. \tag{3.11}$$

Proof. Assume that M is an almost paracontact metric manifold which satisfies (3.11). Then, we can use Lemma 3.6. We prepare,

$$\begin{aligned} N^{(2)}(Y,Z) &= \big(\mathcal{L}_{\varphi Y}\eta\big)(Z) - \big(\mathcal{L}_{\varphi Z}\eta\big)(Y) \\ &= \varphi Y\big(\eta(Z)\big) - \eta\big([\varphi Y, Z]\big) - \varphi Z\big(\eta(Y)\big) + \eta\big([\varphi Z, Y]\big) \\ &= 2\mathrm{d}\eta(\varphi Y, Z) - 2\mathrm{d}\eta(\varphi Z, Y), \end{aligned} \tag{3.12}$$

where LX denotes the Lie derivative with respect to the vector field X. Since Nφ = 0 then,

$$N^{(1)}(Y,Z) = -2\mathrm{d}\eta(Y,Z)\xi. \tag{3.13}$$

Knowing that for all X,Y,Z vector fields on M

$$2(\eta\wedge\omega)(X,Y) = \omega(X)\eta(Y) - \omega(Y)\eta(X), \tag{3.14}$$

the proof is direct, just replace formulas (3.11) and (3.12)-(3.14) in (3.6) we get the formula (3.8), i.e. M is a parCorner manifold.
Conversely, assuming

$$(\nabla_X\varphi)Y = -\eta(X)\big(\eta(Y)\varphi\psi + \omega(\varphi Y)\xi\big).$$

With the help of (3.3), we have

$$\begin{aligned} \mathrm{d}\eta(X,Y) &= \frac{1}{2}\big(g(\nabla_X\xi, Y) - g(\nabla_Y\xi, X)\big) \\ &= (\eta\wedge\omega)(X,Y). \end{aligned}$$

Using (3.3), we get

$$\begin{aligned} 3\mathrm{d}\phi(X,Y,Z) &= X\big(\phi(Y,Z)\big) + Y\big(\phi(Z,X)\big) + Z\big(\phi(X,Y)\big) \\ &\quad -\phi([X,Y],Z) - \phi([Y,Z],X) - \phi([Z,X],Y) \\ &= g\big((\nabla_X\varphi)Y, Z\big) + g\big((\nabla_Y\varphi)Z, X\big) + g\big((\nabla_Z\varphi)X, Y\big), \end{aligned}$$

and using the hypothesis, we obtain

$$\mathrm{d}\phi = 0.$$

To show that (φ, ξ, η, g) is an integrable structure, we rewrite the tensor N_φ with the help of the LeviCivita connection as

$$N_\varphi(X,Y) = \varphi(\nabla_X\varphi)Y - (\nabla_{\varphi X}\varphi)Y - \varphi(\nabla_Y\varphi)X + (\nabla_{\varphi Y}\varphi)X.$$

Using the hypothesis, we get $N_\varphi = 0$. This completes the proof.

The remainder of this paper focuses on the study of the three-dimensional dimension where ψ is a unit space-like vector field.

That is,

$$g(\xi,\xi)=g(\psi,\psi)=-g(\varphi\psi,\varphi\psi)=1, \quad g(\xi,\psi)=g(\xi,\varphi\psi)=g(\psi,\varphi\psi)=0.$$

This formulation readily establishes $\{\xi,\psi,\varphi\psi\}$ as a special ϕ-basis. This set is commonly referred to as the "fundamental basis". As long as the three vector fields for this basis are global, the manifold is parallelizable, this means that its tangent bundle is trivial (that is, isomorphic to the product $M \times R^3$).

In this case, we give two elegant characterizations of 3-dimensional paracorner manifolds as follows:

Theorem 3.5. A 3-dimensional almost paracontact metric structure (φ,ξ,η,g) is a paracorner manifold if and only if

$$\nabla_X\xi=\eta(X)\psi, \tag{3.15}$$

or equivalently,

$$\nabla_{\varphi X}\xi=0. \tag{3.16}$$

for all X vector field on M.
Proof. The necessity was observed in Corollary 3.3. For the sufficiency, suppose that $\nabla_X\xi = \eta(X)\psi$, for all X vector field on M.

In, the author proved that, for an arbitrary 3-dimensional almost paracontact metric manifold (M, φ, ξ, η, g), we have

$$(\nabla_X\varphi)Y=g(\varphi\nabla_X\xi,Y)\xi-\eta(Y)\varphi\nabla_X\xi. \tag{3.17}$$

So, by replacing $\nabla_X\xi$ in (3.17) we obtain (3.8) [14].

For the second formula, it is sufficient to prove that it is equivalent to the first one. Suppose that $\nabla_X\xi = \eta(X)\psi$, so by (3.2) it is easy to see that

$\nabla_{\varphi X}\xi=0.$

Conversely, suppose that $\nabla_{\phi X}\xi = 0$ and replacing X by φX using the formula $\varphi^2X = X - \eta(X)\xi$, we obtain $\nabla_X\xi = \eta(X)\psi$. This completes the proof.

Proposition 3.6. Let (M, φ, ξ, η, g) be a 3-dimensional unit paracorner manifold. Then

1) $\phi=-2\omega\wedge(\omega\circ\varphi)$.
2) $\mathrm{d}\omega=\eta\wedge\nabla_\xi\omega$.

Proof. 1) Using the fact that {ξ,ψ,φψ} is an orthonormal frame, we have

$$\begin{aligned}\varphi X &= g(\varphi X,\xi)\xi+g(\varphi X,\psi)\psi+g(\varphi X,\varphi\psi)\varphi\psi\\ &= \omega(\varphi X)\psi+\omega(X)\varphi\psi.\end{aligned}$$

Thus,

$$\begin{aligned}\phi(X,Y) &= g(\varphi X,Y)\\ &= \omega(\varphi X)\omega(Y)-\omega(X)\omega(\varphi Y)\\ &= -2\big(\omega\wedge(\omega\circ\varphi)\big)(X,Y).\end{aligned}$$

2) We have

$$\mathrm{d}\eta=\eta\wedge\omega. \tag{3.18}$$

Taking the exterior differential, one gets

$$\eta\wedge\mathrm{d}\omega=0, \tag{3.19}$$

hence

$$(\eta \wedge \mathrm{d}\omega)(\xi, X, Y) = 0,$$

we get

$$\mathrm{d}\omega(X,Y) = \eta(X)\mathrm{d}\omega(\xi, Y) + \eta(Y)\mathrm{d}\omega(X, \xi).$$

Moreover, also applying (3.15), one has

$$\begin{aligned}
\mathrm{d}\omega(X,\xi) &= \frac{1}{2}\big((\nabla_X\omega)\xi - (\nabla_\xi\omega)X\big) \\
&= -\frac{1}{2}\big(\eta(X)\psi + (\nabla_\xi\omega)X\big).
\end{aligned}$$

Then, substituting in the previous formula, one gets the second equation in the statement.

Using the fundamental frame, one can get the following:
Proposition 3.7. For any 3-dimensional unit paracorner manifold, we have

$$\begin{array}{lll}
\nabla_\xi\xi = \psi, & \nabla_\psi\xi = 0, & \nabla_{\varphi\psi}\xi = 0, \\
\nabla_\xi\psi = -\xi - \kappa\varphi\psi, & \nabla_\psi\psi = 0, & \nabla_{\varphi\psi}\psi = \alpha\varphi\psi, \\
\nabla_\xi\varphi\psi = \kappa\psi, & \nabla_\psi\varphi\psi = 0, & \nabla_{\varphi\psi}\varphi\psi = \alpha\psi.
\end{array}$$

where $\kappa = g(\nabla_\xi\psi,\varphi\psi)$ and $\alpha = 1 + \mathrm{div}\psi$.

Proof. First of all, since $\{\xi,\psi,\varphi\psi\}$ is a ϕ-basis then for all X vector field on
M we have

$$X = g(X,\xi)\xi + g(X,\psi)\psi - g(X,\varphi\psi)\varphi\psi. \tag{3.20}$$

The first three relations come from formula (3.15). For the fourth relation, since $\{\xi, \psi, \varphi\psi\}$ is an orthonormal frame then,

$$\nabla_\xi\psi = g(\nabla_\xi\psi,\xi)\xi - g(\nabla_\xi\psi,\varphi\psi)\varphi\psi.$$

Using (3.15), we have

$$g(\nabla_\xi\psi,\xi) = -g(\psi,\nabla_\xi\xi) = -1.$$

Then

$$\nabla_\xi\psi = -\xi - \kappa\varphi\psi \qquad \textit{where} \qquad \kappa = g(\nabla_\xi\psi,\varphi\psi).$$

To compute $\nabla_\psi\psi$, we have

$$\nabla_\psi\psi = g(\nabla_\psi\psi,\xi)\xi - g(\nabla_\psi\psi,\varphi\psi)\varphi\psi.$$

Using (3.15), we get

$$g(\nabla_\psi\psi,\xi) = -g(\psi,\nabla_\psi\xi) = 0.$$

For $g(\nabla_\psi\psi,\varphi\psi)$, using Proposition 3.6, we get

$$\begin{aligned}
2\mathrm{d}\omega(\varphi\psi,\psi) &= -g(\nabla_\psi\psi,\varphi\psi) \\
&= 2(\eta\wedge\nabla_\xi\psi)((\varphi\psi,\psi) \\
&= 0,
\end{aligned}$$

then, $\nabla_\psi\psi = 0$. For the sixth relation, we have

$$\nabla_{\varphi\psi}\psi = g(\nabla_{\varphi\psi}\psi,\xi)\xi - g(\nabla_{\varphi\psi}\psi,\varphi\psi)\varphi\psi.$$

Using (3.15), we have $g(\nabla_{\varphi\psi}\psi,\xi) = 0$. For $g(\nabla_{\varphi\psi}\psi,\phi\psi)$ we have

$$\begin{aligned}\mathrm{div}\psi &= g(\nabla_\xi \psi, \xi) - g(\nabla_{\varphi\psi}\psi, \varphi\psi)\\ &= -g(\psi, \nabla_\xi \xi) - g\big(\nabla_{\varphi\psi}\psi, \varphi\psi\big)\\ &= -1 - g(\nabla_{\varphi\psi}\psi, \varphi\psi)\end{aligned}$$

then,

$$g(\nabla_{\varphi\psi}\psi, \varphi\psi) = -1 - \mathrm{div}\psi.$$

Substituting into the above equation we find $\nabla_{\varphi\psi}\psi = (1 + \mathrm{div}\psi)\varphi\psi$. For the remaining relations, just use formulas

$$\nabla_X \varphi Y = (\nabla_X \varphi)Y + \varphi \nabla_X Y,$$

and (3.8) with the obtained relations.

Now, we denote by R the curvature tensor, which is defined for all X,Y,Z $\in\mathfrak{X}$(M) by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]}Z. \tag{3.21}$$

Using Proposition 3.7, standard calculations yield (3.21)

$$\begin{cases} R(\xi, \psi)\xi = \psi,\\ R(\xi, \varphi\psi)\xi = -\alpha\varphi\psi,\\ R(\psi, \varphi\psi)\xi = 0,\\ R(\psi, \varphi\psi)\psi = \big(\alpha^2 + \psi(\alpha)\big)\varphi\psi. \end{cases} \tag{3.22}$$

It is easy to see that

$$R(X, Y)Z = -g(Y, Z)X + g(X, Z)Y,$$

for all X,Y,Z ∈{ξ,ψ,ϕψ} (under the condition $\alpha = -1$). This means that the sectional curvature of a 3-dimensional unit paracorner manifold with α = −1 is c = −1.
The Ricci tensor S, which is defined for all X,Y ∈X(M) by

$$S(X, Y) = g\big(R(X, \xi)\xi, Y\big) + g\big(R(X, \psi)\psi, Y\big) - g\big(R(X, \varphi\psi)\varphi\psi, Y\big),$$

is completely described by

$$\begin{cases} S(\xi, \xi) = \alpha - 1,\\ S(\psi, \psi) = -1 - \alpha^2 - \psi(\alpha),\\ S(\varphi\psi, \varphi\psi) = \alpha(\alpha - 1) + \psi(\alpha),\\ S(\xi, \psi) = S(\xi, \varphi\psi) = S(\psi, \varphi\psi) = 0. \end{cases} \tag{3.23}$$

Considering the signature of the metric from (3.23), the scalar curvature r should be computed as

$$r = S(\xi, \xi) + S(\psi, \psi) - S(\varphi\psi, \varphi\psi) = -2\big(1 + \alpha + \alpha^2 + \psi(\alpha)\big).$$

Class of Examples
Let us denote (x,y,z) the Cartesian coordinates in M = R^3. Let (ei) be the set of vector fields on M defined by:

$$e_1 = \frac{1}{\rho(y, z)}\frac{\partial}{\partial x}, \qquad e_2 = \frac{1}{\tau(y, z)}\frac{\partial}{\partial y}, \qquad e_3 = \frac{1}{\sigma(y, z)}\frac{\partial}{\partial z},$$

where ρ,τ and σ are three functions on M non zero every where, such that

$$g(e_1, e_1) = g(e_2, e_2) = -g(e_3, e_3) = 1, \quad g(e_1, e_2) = g(e_1, e_3) = g(e_2, e_3) = 0.$$

That is, the form of the metric g becomes:

$$g = \begin{pmatrix} \rho(y,z)^2 & 0 & 0\\ 0 & \tau(y,z)^2 & 0\\ 0 & 0 & -\sigma(y,z)^2 \end{pmatrix}.$$

Further, we define an almost contact metric (ϕ,ξ,η) on R^3 by

$$\varphi = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{\sigma(y,z)}{\tau(y,z)} \\ 0 & \frac{\tau(y,z)}{\sigma(y,z)} & 0 \end{pmatrix}, \quad \xi = e_1, \quad \eta = \rho(y,z)\, dx.$$

One can easily check that

$$\eta(\xi) = 1, \quad \varphi^2 e_i = e_i - \eta(e_i)\xi \quad and \quad g(\varphi e_i, \varphi e_j) = -g(e_i, e_j) + \eta(e_i)\eta(e_j),$$

That means (ϕ,ξ,η,g) is an almost paracontact structure on M.

By Koszul's formula, the components of the Levi-Civita connection corresponding to g are given by

$$\begin{array}{lll} \nabla_{e_1} e_1 = -\frac{\rho_2}{\rho\tau} e_2 + \frac{\rho_3}{\rho\sigma} e_3, & \nabla_{e_1} e_2 = \frac{\rho_2}{\rho\tau} e_1, & \nabla_{e_1} e_3 = \frac{\rho_3}{\rho\sigma} e_1, \\ \nabla_{e_2} e_1 = 0, & \nabla_{e_2} e_2 = \frac{\rho_3}{\rho\sigma} e_3, & \nabla_{e_2} e_3 = \frac{\tau_3}{\tau\sigma} e_2, \\ \nabla_{e_3} e_1 = 0, & \nabla_{e_3} e_2 = \frac{\sigma_2}{\tau\sigma} e_3, & \nabla_{e_3} e_3 = \frac{\sigma_2}{\tau\sigma} e_2, \end{array}$$

where $\rho_i = \frac{\partial \rho(y,z)}{\partial x_i}$ and $\sigma_i = \frac{\partial \sigma(y,z)}{\partial x_i}$ with $i \in \{1,2\}$.

According to Theorem 3.5, (M, ϕ, ξ, η, g) is a two parameters family of paracorner manifolds with

$$\psi = -\frac{\rho_2}{\tau\rho} e_2 + \frac{\rho_3}{\rho\sigma} e_3.$$

To obtain a unit paracorner manifold i.e. g(ψ,ψ) = 1, it is enough to take

$$\rho = \sigma, \quad \rho_3 = 0 \quad and \quad \tau = -\frac{\rho_2}{\rho}.$$

Thus we obtain

$$\nabla_{e_1} e_1 = e_2, \quad \nabla_{e_1} e_2 = -e_1, \quad \nabla_{e_3} e_2 = -e_3, \quad \nabla_{e_3} e_3 = -e_2, \quad \text{others} = 0,$$

And

$$\psi = e_2, \qquad \omega = -\frac{\rho_2}{\rho} dy, \qquad \text{div}\psi = -2 \quad and \quad \kappa = 0.$$

Also, in this case we have

$$R(e_1, e_2)e_1 = e_2, \; R(e_1, e_3)e_1 = e_3, \; R(e_2, e_3)e_1 = 0, \; R(e_2, e_3)e_2 = e_3,$$

And

$$S(e_1, e_1) = S(e_2, e_2) = -S(e_3, e_3) = -2 \qquad and \qquad \text{others} \; = 0.$$

This information confirms the results in Proposition 3.7 and (3.22)-(3.23).

Generalized Ricci-Yamabe Soliton on 3-Dimensional Unit Paracorner Manifold

As we mentioned in the introduction, we will present here the concept of generalized Ricci-Yamabe soliton with proof of existence on the 3-dimensional unit paracorner manifold.

Definition 4.1. A generalized Ricci-Yamabe soliton (briefly, GRYS) is defined as a semi-Riemannian manifold (M,g) equipped with a global vector field V on M that satisfies

$$\mathcal{L}_V g = -2c_1 V^\flat \otimes V^\flat + 2c_2 S + 2(\lambda + \mu r) g. \tag{4.1}$$

First, note that for a three-dimensional unit paracorner manifold, ξ and ϕψ are Killing vector fields that is

$$\mathcal{L}_\xi g = \mathcal{L}_{\varphi\psi} g = 0.$$

Then, generalized Ricci-Yamabe soliton with potential vector field ξ or ψ is trivial. So, let us take ψ as a potential vector field and prove the existence of a non-trivial generalized Ricci-Yamabe soliton on three-dimensional unit paracorner manifolds.

With the help of Proposition 3.7, the components of the Lie derivative Lψg are given by:

$$\begin{cases} (\mathcal{L}_\psi g)(\xi,\xi) = -2, \\ (\mathcal{L}_\psi g)(\xi,\psi) = 0, \\ (\mathcal{L}_\psi g)(\xi,\varphi\psi) = -\kappa, \\ (\mathcal{L}_\psi g)(\psi,\psi) = 0, \\ (\mathcal{L}_\psi g)(\psi,\varphi\psi) = 0, \\ (\mathcal{L}_\psi g)(\varphi\psi,\varphi\psi) = -2\alpha. \end{cases} \tag{4.2}$$

Now, substituting (4.2), (3.23) with $r = -2\big(1+\alpha+\alpha^2+\psi(\alpha)\big)$ in (4.1), one can get the following system:

$$\begin{cases} c_2(\alpha-1)+\lambda+\mu r = -1, \\ \kappa = 0, \\ -c_1 - c_2\big(1+\alpha^2+\psi(\alpha)\big)+\lambda+\mu r = 0, \\ \alpha + c_2\big(\alpha^2-\alpha+\psi(\alpha)\big)-\lambda-\mu r = 0. \end{cases} \tag{4.3}$$

From the first equation directly we get

$$\lambda = c_2(1-\alpha)-\mu r - 1. \tag{4.4}$$

By adding the third and fourth equations of the above system, we find:

$$c_1 = \alpha - c_2(\alpha+1). \tag{4.5}$$

By substituting for (4.4) and (4.5) in the third equation of the system, we find

$$\alpha + 1 = c_2\big(1-\alpha^2-\psi(\alpha)\big). \tag{4.6}$$

By summarizing this discussion, we obtain the following theorem:

Theorem 4.2. Let (M,ϕ,ξ,η,g) be a three dimensional unit paracorner manifold with κ = 0. Generalized Ricci-Yamabe soliton equation

$$\mathcal{L}_\psi g = -2c_1\omega\otimes\omega + 2c_2 S + 2(\lambda+\mu r)g,$$

admits the following solutions on 3-dimensional unit paracorner manifold:

$$c_1 = \alpha - c_2(\alpha+1), \qquad \alpha+1 = c_2\big(1-\alpha^2-\psi(\alpha)\big),$$
$$\lambda = -1 + c_2(1-\alpha) + 2\mu\big(1+\alpha+\alpha^2+\psi(\alpha)\big) \quad and \quad c_2,\mu\in\mathbb{R}.$$

By analyzing this theorem, we discover two interesting cases:
First, for $\alpha = 0$ we get

$$c_1 = -c_2 = -1, \quad \lambda = 2\mu.$$

Second, for $\alpha = -1$ we obtain

$$c_1 = -1, \quad \lambda = -1 + 2c_2 + 2\mu \quad and \quad c_2\in\mathbb{R}.$$

Example 4.3. For a non-trivial example, let us use the example above. For $\rho=\sigma, \quad \rho_3 = 0 \quad and \quad \tau = -\frac{\rho_2}{\rho}$, we have proved that (M,ϕ,ξ,η,g) is a unit paracorner manifold with

$$\psi = e_2, \qquad \omega = -\frac{\rho_2}{\rho}dy, \qquad \mathrm{div}\psi = -2 \quad and \quad \kappa = 0.$$

and

$$S(e_1,e_1) = S(e_2,e_2) = -S(e_3,e_3) = -2, \quad others\ = 0 \quad and \quad r = -2.$$

Thus, with simple computations we obtain

$$\mathcal{L}_\psi g(e_1, e_1) = -\mathcal{L}_\psi g(e_3, e_3) = -2, \qquad others \ = 0.$$

According to the second case in the above Theorem (since $\alpha = -1$), we can see that the equation

$$\mathcal{L}_\psi g = 2\omega \otimes \omega + 2c_2 S + 2(2c_2 - 1)g,$$

is satisfied for $c_2 \in \mathbb{R}$.

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