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Put Call Duality for the Truncated Normal Distribution Model in Option Pricing

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Abstract

This study explored the put-call duality under the Truncated Normal Distribution (TND) model in option pricing, which assumed that the log-prices of the underlying assets were bounded above and below. The model was compared with the famous Black-Scholes Model, which has consistently been considered the benchmark model for option pricing. Using the model, we derived a model-consistent put-call duality identity. We applied it to construct an objective function to improve the accuracy and consistency of option pricing. This is in conjunction with the various modifications put forward to "improve" the pricing of options. This study extended Zhu & He's work by incorporating the put-call duality relation in the proposed model and compared the results to those obtained using the classical Black & Scholes (BS) Model. Similarly, this study investigated the implied volatility of the proposed model and compared it with the BS model. From the duality constraint calibration, the proposed model performed significantly better than the traditional Black-Scholes model. A numerical analysis was done using S&P 500 options data retrieved on June 27, 2025. Model performance checks, such as AIC, BIC, and pricing errors, indicated the outstanding performance of the TND model. From the methodology and numerical analysis, there was evidence that the duality relation holds for the Truncated Normal Distribution model, as is the case with the Black & Scholes model. The implied volatility indicated that the TND model was sensitive to in-and-out-of-the-money options. Therefore, enforcing the duality constraint in the pricing of European options improved the market consistency, ensuring that no arbitrage opportunities exist in the market that can provide enormous risk-free earnings.

Keywords: Put-Call Duality, Truncated Normal Distribution, Implied Volatility, Martingale Restriction, Delta Hedging

Introduction

In modern financial markets, derivative pricing models play a central role in risk management, trading strategies, and valuation. Financial instruments such as options, warrants, swaps and futures are widely used in financial market. Of these instruments, options are considered as the mostly used and traded tools. These financial instruments (options) which derive their prices from an underlying asset (s) have attracted the attention of investor and researcher on the optimal way of pricing them. Options have their origins since the 1600s - where they was used by the Dutch for the harvesting of olives, Romans and Phoenicians for the transportation of cargo [1]. They have been known to be the most preferred tools used in the market by investor to hedge against unforeseeable risk.

To protect themselves from the extreme losses, investors adopt the usage of market order and trading orders. Trade orders are orders which are placed by investors to either buy or sell a stock when a specified price is reached or a stock takes a specific direction as specified by the investor. For instance, stop limit orders are exercised when a specified price is reached by the stock while market orders are exercised whenever they are placed at current market price. Investors may specify a percentage for which if the price fluctuates beyond, their orders are exercised. The choice of the prices for which the orders are set and which type of order to choose posses a dilemma to investors are indicated by Bae et al. (2003) [2]. The use of trade orders has resulted in the truncation of the stock prices thereby providing the bounded prices - hence the crucial role of the TND model.

The put-call duality is a fundamental concept in option pricing theory, providing a symmetry between call and put option prices when viewed under transformations of market parameters such as the current stock price of an underlying assets and the option's strike price. While this duality is well-understood in classical models - most notably the Black-Scholes framework in Alfonsi and Jourdain (2006, 2009), Carr and Bowie (1994), Carr et al. (2002), Chesney and Jeanblanc (2004), and Swishchuk (2023), which assumes log-normal price dynamics - its extension to more realistic, bounded, and non-Gaussian models remains relatively under-explored [3-8]. Additionally, although recent research has explored the application of the TND in option pricing and calibration to market data M. Cox (2009), del Castillo and Daoudi (2009), Hasan et al. (2012), Norgaard and Killeen (1980), Onsoti (2020), and Zhu and He (2018), the theoretical underpinnings of put-call duality within this model remain underdeveloped [9-14]. This gap in the literature prompts a rigorous investigation of duality relationships within the TND framework. This paper develops and analyzes a formulation of put-call duality for the TND model. Specifically, we derived the martingale restrictions and dual pricing identities that are critical under the TND model assumption and hence must be satisfied. We assessed the conditions under which the classical duality relations persist, and examined the modifications necessitated by the presence of truncation bounds of asset returns. The results contribute not only to the theoretical understanding of dual pricing under distributional constraints, but also offer insights for practical applications in model calibration, hedging strategies, and market consistency checks. Additionally, this paper investigated the implied volatility of the proposed model follows the analogy of the BSM model. A numerical example presented in this study helps to affirm this argument.

The structure of the paper is as follows: Section 2 reviews the related literature of the put-call duality and introduces the TND-based option pricing model. Model calibration of the duality conditions under the truncated normal assumption, with emphasis on the martingale property and change-of-measure implications is presented in section 3. Section 4 presents practical implication of the proposed model while section 5 provides numerical analysis, results, discussions and the implied volatility depicted by the behavior of dual pricing. Section 6 concludes with model implications, recommendations, and future research.

Related Literature

Recent developments in financial modeling have emphasized the limitations of the log-normal assumption in asset pricing, particularly in its inability to capture empirical features of asset returns such as bounded support, skewness, and excess kurtosis. These limitations have motivated the use of alternative distributions to model asset returns of the underlying asset and the assumption of constant volatility. For instance, Merton (1973) presented an extension of the Black & Scholes model by pricing options when the underlying assets are discontinuous [15]. On the other hand, Dumas et al. (1998) studied the volatility of the asset price in relation to time and proposed the famous volatility smile [16]. The introduction of the truncated Lévy process in option pricing was done by Brechmann et al. (2012) [17]. Additionally, while investigating the underlying asset's return, Peiro, 1999; Rachev et al., 2005 found out that the returns were actually skewed and fat-tailed respectively [18,19]. Similar results were obtained by Mwaniki, 2019 that the log-returns of the underlying asset are heavy-tailed which is not in line with the normality assumption proposed by Black & Scholes [20]. The constant volatility assumption and normal distribution of the underlying asset among other parameter assumptions have been investigated by academic researchers. In many of these studies, the return of the underlying asset is assumed to be normally distributed with no bounds. Further, Basnarkov et al. (2019) while investigating the heavy-tailed distribution of log-return in option pricing found that social behaviors significantly influenced the stock market [21]. Further, the study suggested that extreme market events reshape market expectations and risk assessment though infrequent.

Furthermore, Zhu and He (2018) proposed a bounded normal distribution - Truncated Normal Distribution (TND) for modeling the log-return of the underlying asset [14]. This model presents a compelling framework by explicitly restricting asset prices to a finite interval, thereby reflecting realistic price bounds and avoiding undesirable tail behavior. The TND has been applied across various disciplines. For instance, Norgaard and Killer applied the truncated normal distribution to analyze investment [12]. In this study, Norgaad found out that the truncated normal distribution serves as good alternative in when investors are not contended that the probability at the tail end of normal distribution is not in line with their investment decisions. Also Hasan et al. (2012) and Dey and Chakraborty (2012) used the truncated normal distribution in measuring the efficiency of the stock market and modeling of the single period inventory model respectively [11,22]. Hasan found out that the truncated normal distribution was in a better position for measuring the technical inefficiency in Bangladesh Stock Market as opposed to half-normal distributions. Similarly, Dey reported that the truncated normal distribution provided higher profits than the non-truncated case. Other studies where the truncated normal distribution has been applied is the queuing process of the impatient customer and in the investigation of portfolio insurance by Pender (2015) and Hocquard et al. (2015) respectively [23,24]. Pender (2015) concluded that the truncated normal distributions approximate the mean, variance and kurtosis of the queue process better than the normal distribution although the skewness was overestimated [23]. On the hand, Hocquard reported that the left truncated normal distribution included into the Payoff Distribution Model reduces the downside risk for the investor significantly with no requirement of the fund manager. The left-truncated model minimizes the downward risk inherent in the investment portfolio when the normal distribution is used.

Similarly, Gatti et al. (2003) introduced the truncated normal distribution in analyzing the pattern of entry and exit of firms in industrial market based on the financial bankruptcy - more specifically the firm's accumulation of capital [25]. Gatti found that the entry and exit of firms follows the truncated normal distribution and that firms that remain in the market posses a higher equity ratio and low volatility (right-truncation) while firms that exit the market experiences weak equity ratio or have higher fragility in bankruptcy (left-truncation). Furthermore, del Castillo and Daoudi (2009) used the mixture of left-right truncated normal distribution in the pricing of the bid and ask price of the exchange rates between the Euros and dollars [10]. From the findings of the study, Del found that the truncated normal distribution (left and right) performed better than other distribution such as Laplace, inverse Gaussian, normal and the mixture of normal distribution is measuring the exchange rates between the US dollar and European euros. Additionally, M. Cox (2009) incorporated the truncated normal distribution in the development of the statistical process control - commonly referred to as control charts and their application in finance [9]. The findings of the study indicated that truncated financial data provides reliable charts that show better fit with predicable behaviors that are quite reliable. The TND model was also applied in the pricing of European options by Onsoti (2020) in the calibration of Facebook, Apple and Russel 2000 index options [13]. Furthermore, Cao et al. (2023) calibrated a model for pricing European option whereby the underlying asset returns followed a logarithmic truncated t-distribution [26]. As noted by Fajardo and Mordecki (2006) and Eberlein et al. (2008), the put-call duality relationship is a result of the "change of the risk-neutral probability measure" [27,28]. The duality relationship has been applied across various models such as Carr and Bowie (1994), Carr et al. (2002), Chesney and Jeanblanc (2004), among others. Additionally, Albrecher et al. (2007) highlighted the implications of the duality relationship in option pricing when calibrating models in a truncated Lévy market [5-7,29]. Understanding the structure of the existing and proposed pricing models is of key interest in option pricing. For instance, the relationship between the call and put prices is important for hedging and risk minimization purposes.

Put-call parity and risk-neutral pricing has been extensively applied in risk management to provide a framework for investors to appropriately value options in the face of market uncertainties. According to Taleb (2014), the intrinsic relationship between puts and calls, enforced by put-call parity, supports a structured evaluation of expected terminal payoffs [30]. This allows investors and traders to employ a mean that aligns with the forward price, streamlining the decision-making process in environments characterized by incomplete markets and operational challenges of dynamic hedging. As traditional hedging methods are not feasible, employing this simplified framework guided by risk-neutral principles helps investors manage the inherent volatility in financial instruments [30]. Furthermore, the integration of truncated normal distributions into option pricing expands the strategies for risk management in trading. As suggested by Basnarkov et al. (2019), the application of truncated distributions not only stabilizes pricing but also ensures that the emergence of arbitrage opportunities are minimized [21]. Their findings indicated that option prices derived from distributions with truncated tails demonstrate robustness against fluctuations, further emphasizing the model's utility in generating fair pricing over discrete time intervals. This is particularly pertinent in illustrating how alternative pricing pathways can enhance predictive accuracy, providing an advantageous complement to traditional such as BS model [31].

The fusion of put-call duality concepts with truncated normal distribution models offers novel avenues for managing risk in option pricing. Both the structured approach of risk-neutral pricing and the stability attained through truncated distributions equip investors with effective strategies to navigate market complexities, fostering better decision-making and enhancing the reliability of option valuation. By acknowledging these methodologies, investors and traders of financial instruments can better adapt to the multifaceted nature of option pricing in contemporary markets.

Methodology

Mathematical Preliminaries

Consider a random variable X which is assumed to follow a truncated normal distribution with mean, $\theta \in R$, variance, γ^2, $(\gamma > 0)$, and $x \in [\alpha,\beta]$, then its probability density function (p.d.f) is given by

$$f(x;\gamma,\theta,\alpha,\beta) = \begin{cases} \frac{1}{\gamma}\frac{\psi\left(\frac{x-\theta}{\gamma}\right)}{\Psi\left(\frac{\beta-\theta}{\gamma}\right)-\Psi\left(\frac{\alpha-\theta}{\gamma}\right)}, & \alpha \le x \le \beta \\ 0, & \text{otherwise} \end{cases}$$

where: $\psi(*)$ and $\Psi(*)$ are the standard normal density and distribution functions respectively.
Assuming that the log-returns of the underlying asset prices follows a truncated normal distribution as noted by Zhu and He (2018), conditioned on the filtration of the current asset price, i.e. [14].

$$E^{\mathbb{Q}}\left[e^{-r\tau}S_t\middle|\mathcal{F}_0\right] = S_0 \tag{1}$$

under the probability measure Q, then the random variable X, which has a p.d.f, $f(x;\theta\tau,\gamma\sqrt{\tau},\alpha,\beta)$, can be defined as;

$$X_t = \ln\left(\frac{S_t}{S_0}\right) = \ln(Z_t) \quad \text{where} \quad Z_t = \left(\frac{S_t}{S_0}\right)$$
$$\Rightarrow Z_t = e^{X_t}$$

Therefore, using the change of variable technique, the probability density function of the log-price of the underlying asset can easily be determined as follows;

$$f_Z(z) = \begin{cases} \frac{1}{\gamma\sqrt{\tau}z}\frac{\psi\left(\frac{\log_e(z)-\theta t}{\gamma\sqrt{\tau}}\right)}{\Psi\left(\frac{\beta-\theta\tau}{\gamma\sqrt{\tau}}\right)-\Psi\left(\frac{\alpha-\theta\tau}{\gamma\sqrt{\tau}}\right)}, & e^{\alpha} \le z \le e^{\beta} \\ 0, & \text{otherwise} \end{cases}$$

According to Zhu and He (2018), the call of a European Option is given by; [14]

$$C(\tau,S_0)^{TND} = S_0\left[\frac{\Psi\left(\lambda-\gamma\sqrt{\tau}\right)-\Psi\left(\nu-\gamma\sqrt{\tau}\right)}{\Psi\left(\lambda-\gamma\sqrt{\tau}\right)-\Psi\left(\varphi-\gamma\sqrt{\tau}\right)}\right] - Ke^{r\tau}\left[\frac{\Psi(\lambda)-\Psi(\nu)}{\Psi(\lambda)-\Psi(\varphi)}\right] \tag{2}$$

While the price of a European option put is given by:

$$P(\tau, S_0)^{TND} = Ke^{r\tau}\left[\frac{\Psi(\nu) - \Psi(\varphi)}{\Psi(\lambda) - \Psi(\varphi)}\right] - S_0\left[\frac{\Psi\left(\nu - \gamma\sqrt{\tau}\right) - \Psi\left(\varphi - \gamma\sqrt{\tau}\right)}{\Psi\left(\lambda - \gamma\sqrt{\tau}\right) - \Psi\left(\varphi - \gamma\sqrt{\tau}\right)}\right] \quad (3)$$

$$\text{where} \quad \lambda = \frac{\beta - \theta\tau}{\gamma\sqrt{\tau}}, \quad \nu = \frac{\ln(\frac{K}{S_0}) - \theta\tau}{\gamma\sqrt{\tau}}, \quad \text{and} \quad \varphi = \frac{\alpha - \theta\tau}{\gamma\sqrt{\tau}}$$

Duality Relation Using the Classical Black & Scholes (BS) Model

Proposition 1. Let S_t be the price of the underlying stock/asset at time t, K be the option's strike price, r be the risk - free interest rate, γ be the volatility of the underlying stock/asset, τ be the time to maturity of the option, then the put-call duality of a European Call under the BS model be given by:

$$C(t, -S_t)^{BS} = S_t\Psi(d_1) - Ke^{-r\tau}\Psi(d_2) + \left[Ke^{-r\tau} - S_t\right] \qquad \textit{where,}$$

$$d_1 = \frac{\ln\left(\frac{S_t}{K}\right) + \left(r + \frac{\gamma^2}{2}\right)\tau}{\gamma\sqrt{\tau}}$$

$$d_2 = d_1 - \gamma\sqrt{\tau}$$

and $\Psi(*)$ be the standard normal cumulative density function (CDF)

Proof. : The European call and put option under for the BS model for a non-dividend paying is given by;

$$C_t = S_t\Psi(d_1) - Ke^{-r\tau}\Psi(d_2) \quad (4)$$

$$P_t = Ke^{-r\tau}\Psi(-d_2) - S_t\Psi(-d_1) \quad (5)$$

where

$$d_1 = \frac{\ln\left(\frac{S_t}{K}\right) + \left(r + \frac{\gamma^2}{2}\right)\tau}{\gamma\sqrt{\tau}}$$

$$d_2 = d_1 - \gamma\sqrt{\tau}$$

According to Peskir and Shiryaev (2002) and by defining $\tilde{S}_t = -S_t$, $\tilde{K} = -K$ and $\tilde{\gamma} = -\gamma$, then the put call duality for the BS model is given by; [32]

$$C(t, \tilde{S}_t)^{BS} = \tilde{S}_t\Psi\left[\frac{\ln\left(\frac{\tilde{S}_t}{\tilde{K}}\right) + \left(r + \frac{\tilde{\gamma}^2}{2}\right)\tau}{\tilde{\gamma}\sqrt{\tau}}\right] - \tilde{K}e^{-r\tau}\Psi\left[\frac{\ln\left(\frac{\tilde{S}_t}{\tilde{K}}\right) + \left(r + \frac{\tilde{\gamma}^2}{2}\right)\tau}{\tilde{\gamma}\sqrt{\tau}} - \tilde{\gamma}\sqrt{\tau}\right] \quad (6)$$

Defining

$$\tilde{d}_1 = \frac{\ln\left(\frac{\tilde{S}_t}{\tilde{K}}\right) + \left(r + \frac{\tilde{\gamma}^2}{2}\right)\tau}{\gamma\sqrt{\tau}} \qquad \text{and}$$

$$\tilde{d}_2 = \frac{\ln\left(\frac{\tilde{S}_t}{\tilde{K}}\right) + \left(r + \frac{\tilde{\gamma}^2}{2}\right)\tau}{\tilde{\gamma}\sqrt{\tau}} - \tilde{\gamma}\sqrt{\tau} = \tilde{d}_1 - \tilde{\gamma}\sqrt{\tau}$$

Then,

$$C(t, \tilde{S}_t)^{BS} = \tilde{S}_t\Psi(\tilde{d}_1) + \tilde{K}e^{-r\tau}\Psi(\tilde{d}_2)$$

Since $\tilde{S}_t = -S_t$, $\tilde{K} = -K$ and $\tilde{\gamma} = -\gamma$, replacing them in the equation we have

$$\tilde{d}_1 = \frac{\ln\left(\frac{-S_t}{-K}\right) + \left(r + \frac{(-\gamma)^2}{2}\right)\tau}{-\gamma\sqrt{\tau}} = -\frac{\ln\left(\frac{S_t}{K}\right) + \left(r + \frac{\gamma^2}{2}\right)\tau}{\gamma\sqrt{\tau}} = -d_1$$

$$\tilde{d}_2 = -d_1 - (-\gamma)\sqrt{\tau} = -d_2$$

$$\Rightarrow C(t, -S_t)^{BS} = -S_t\Psi(-d_1) + Ke^{-r\tau}\Psi(-d_2)$$

Therefore, the put - call duality under the BS model is given as;

$$\begin{aligned} C(t, -S_t)^{BS} &= Ke^{-r\tau}\Psi(-d_2) - S_t\Psi(-d_1) \\ &= Ke^{-r\tau}[1 - \Psi(d_2)] - S_t[1 - \Psi(d_1)] \\ &= S_t\Psi(d_1) - Ke^{-r\tau}\Psi(d_2) + \left[Ke^{-r\tau} - S_t\right] \end{aligned}$$

$$\therefore C(t, -S_t)^{BS} = S_t\Psi(d_1) - Ke^{-r\tau}\Psi(d_2) + \left[Ke^{-r\tau} - S_t\right] = P(t, S_t)^{BS} \quad (7)$$

Hence, proof.

Duality Relation Using the Truncated Normal Distribution (TND) Model

Proposition 2. Let the underlying asset price S_t at maturity T follow a truncated normal distribution on the interval [α,β] with θ and γ as the location and scale parameters, then the put - call duality of a European Call Option is given by;

$$C^{dual} = Ke^{r\tau}\left[\frac{\Psi(\nu)-\Psi(\varphi)}{\Psi(\lambda)-\Psi(\varphi)}\right] - S_0\left[\frac{\Psi(\nu-\gamma\sqrt{\tau})-\Psi(\varphi-\gamma\sqrt{\tau})}{\Psi(\lambda-\gamma\sqrt{\tau})-\Psi(\varphi-\gamma\sqrt{\tau})}\right] \tag{8}$$

$$\text{where} \quad \lambda = \frac{\beta-\theta\tau}{\gamma\sqrt{\tau}}, \quad \nu = \frac{\ln(\frac{K}{S_0})-\theta\tau}{\gamma\sqrt{\tau}}, \quad \text{and} \quad \varphi = \frac{\alpha-\theta\tau}{\gamma\sqrt{\tau}}$$

Proof. : The call price of a European call option under the truncated normal distribution is given by;

$$C^{dual} = S_0\left[\frac{\Psi\left[\frac{\beta-\theta\tau}{\gamma\sqrt{\tau}}-\gamma\sqrt{\tau}\right]-\Psi\left[\frac{\ln(\frac{K}{S_0})-\theta\tau}{\gamma\sqrt{\tau}}-\gamma\sqrt{\tau}\right]}{\Psi\left[\frac{\beta-\theta\tau}{\gamma\sqrt{\tau}}-\gamma\sqrt{\tau}\right]-\Psi\left[\frac{\alpha-\theta\tau}{\gamma\sqrt{\tau}}-\gamma\sqrt{\tau}\right]}\right] - Ke^{-r\tau}\left[\frac{\Psi\left[\frac{\beta-\theta\tau}{\gamma\sqrt{\tau}}\right]-\Psi\left[\frac{\ln(\frac{K}{S_0})-\theta\tau}{\gamma\sqrt{\tau}}\right]}{\Psi\left[\frac{\beta-\theta\tau}{\gamma\sqrt{\tau}}\right]-\Psi\left[\frac{\alpha-\theta\tau}{\gamma\sqrt{\tau}}\right]}\right] \tag{9}$$

Using the same approach as was with the B-S model above and by substituting S_0,K,β,α and γ by $-S_0$,−K,−β,−α and −γ respectively, the put - call duality under the TND model is determined as follows;

$$\tilde{C}^{dual} = -S_0\left[\frac{\Psi\left[\frac{-\beta-\theta\tau}{(-\gamma)\sqrt{\tau}}-(-\gamma)\sqrt{\tau}\right]-\Psi\left[\frac{\ln\left(\frac{-K}{-S_0}\right)-\theta\tau}{(-\gamma)\sqrt{\tau}}-(-\gamma)\sqrt{\tau}\right]}{\Psi\left(\frac{-\beta-\theta\tau}{(-\gamma)\sqrt{\tau}}-(-\gamma)\sqrt{\tau}\right)-\Psi\left(\frac{-\alpha-\theta\tau}{(-\gamma)\sqrt{\tau}}-(-\gamma)\sqrt{\tau}\right)}\right] - (-K)e^{-r\tau}\left[\frac{\Psi\left[\frac{-\beta-\theta\tau}{(-\gamma)\sqrt{\tau}}\right]-\Psi\left[\frac{\ln\left(\frac{-K}{-S_0}\right)-\theta\tau}{(-\gamma)\sqrt{\tau}}\right]}{\Psi\left(\frac{-\beta-\theta\tau}{(-\gamma)\sqrt{\tau}}\right)-\Psi\left(\frac{-\alpha-\theta\tau}{(-\gamma)\sqrt{\tau}}\right)}\right]$$

$$= -S_0\left[\frac{\Psi\left[\frac{-\beta-\theta\tau}{(-\gamma)\sqrt{\tau}}+\gamma\sqrt{\tau}\right]-\Psi\left[\frac{\ln\left(\frac{K}{S_0}\right)-\theta\tau}{(-\gamma)\sqrt{\tau}}+\gamma\sqrt{\tau}\right]}{\Psi\left[\frac{-\beta-\theta\tau}{(-\gamma)\sqrt{\tau}}+\gamma\sqrt{\tau}\right]-\Psi\left[\frac{-\alpha-\theta\tau}{(-\gamma)\sqrt{\tau}}+\gamma\sqrt{\tau}\right]}\right] + Ke^{-r\tau}\left[\frac{\Psi\left(\frac{-\beta-\theta\tau}{(-\gamma)\sqrt{\tau}}\right)-\Psi\left(\frac{\ln\left(\frac{K}{S_0}\right)-\theta\tau}{(-\gamma)\sqrt{\tau}}\right)}{\Psi\left(\frac{-\beta-\theta\tau}{(-\gamma)\sqrt{\tau}}\right)-\Psi\left(\frac{-\alpha-\theta\tau}{(-\gamma)\sqrt{\tau}}\right)}\right]$$

$$= -S_0\left[\frac{\Psi\left(-\left[\frac{-\beta-\theta\tau}{\gamma\sqrt{\tau}}-\gamma\sqrt{\tau}\right]\right)-\Psi\left(-\left[\frac{\ln\left(\frac{K}{S_0}\right)-\theta\tau}{\gamma\sqrt{\tau}}-\gamma\sqrt{\tau}\right]\right)}{\Psi\left(-\left[\frac{-\beta-\theta\tau}{\gamma\sqrt{\tau}}-\gamma\sqrt{\tau}\right]\right)-\Psi\left(-\left[\frac{-\alpha-\theta\tau}{\gamma\sqrt{\tau}}-\gamma\sqrt{\tau}\right]\right)}\right] + Ke^{-r\tau}\left[\frac{\Psi\left(-\left[\frac{-\beta-\theta\tau}{\gamma\sqrt{\tau}}\right]\right)-\Psi\left(-\left[\frac{\ln\left(\frac{K}{S_0}\right)-\theta\tau}{\gamma\sqrt{\tau}}\right]\right)}{\Psi\left(-\left[\frac{-\beta-\theta\tau}{\gamma\sqrt{\tau}}\right]\right)-\Psi\left(-\left[\frac{-\alpha-\theta\tau}{\gamma\sqrt{\tau}}\right]\right)}\right]$$

By the analogy of the symmetry of the normal distribution, we have that:

$$\begin{aligned}\Psi(-u) &= \int_u^{\infty}\psi(v)dv \\ &= \int_{-\infty}^{\infty}\psi(v)dv - \int_{-\infty}^{u}\psi(v)dv \\ &= 1-\Psi(u)\end{aligned}$$

Therefore, the symmetry of the normal distribution is given by;

$$\Psi(-u) = 1-\Psi(u) \tag{10}$$

Then

$$\tilde{C}^{dual} = -S_0\left[\frac{1-\Psi\left[\frac{-\beta-\theta\tau}{\gamma\sqrt{\tau}}-\gamma\sqrt{\tau}\right]-1+\Psi\left[\frac{\ln(\frac{K}{S_0})-\theta\tau}{\gamma\sqrt{\tau}}-\gamma\sqrt{\tau}\right]}{1-\Psi\left[\frac{-\beta-\theta\tau}{\gamma\sqrt{\tau}}-\gamma\sqrt{\tau}\right]-1+\Psi\left[\frac{-\alpha-\theta\tau}{\gamma\sqrt{\tau}}-\gamma\sqrt{\tau}\right]}\right] + Ke^{-r\tau}\left[\frac{1-\Psi\left[\frac{-\beta-\theta\tau}{\gamma\sqrt{\tau}}\right]-1+\Psi\left[\frac{\ln(\frac{K}{S_0})-\theta\tau}{\gamma\sqrt{\tau}}\right]}{1-\Psi\left[\frac{-\beta-\theta\tau}{\gamma\sqrt{\tau}}\right]-1+\Psi\left[\frac{-\alpha-\theta\tau}{\gamma\sqrt{\tau}}\right]}\right]$$

$$= -S_0\left[\frac{\Psi\left(\frac{\ln(\frac{K}{S_0})-\theta\tau}{\gamma\sqrt{\tau}}-\gamma\sqrt{\tau}\right)-\Psi\left(\frac{-\beta-\theta\tau}{\gamma\sqrt{\tau}}-\gamma\sqrt{\tau}\right)}{\Psi\left(\frac{-\alpha-\theta\tau}{\gamma\sqrt{\tau}}-\gamma\sqrt{\tau}\right)-\Psi\left(\frac{-\beta-\theta\tau}{\gamma\sqrt{\tau}}-\gamma\sqrt{\tau}\right)}\right] + Ke^{-r\tau}\left[\frac{\Psi\left(\frac{\ln(\frac{K}{S_0})-\theta\tau}{\gamma\sqrt{\tau}}\right)-\Psi\left(\frac{-\beta-\theta\tau}{\gamma\sqrt{\tau}}\right)}{\Psi\left(\frac{-\alpha-\theta\tau}{\gamma\sqrt{\tau}}\right)-\Psi\left(\frac{-\beta-\theta\tau}{\gamma\sqrt{\tau}}\right)}\right]$$

Assuming that −α = β, implying that −β = α, we have then that;

$$\tilde{C}^{dual} = -S_0\left[\frac{\Psi\left(\frac{\ln\left(\frac{K}{S_0}\right)-\theta\tau}{\gamma\sqrt{\tau}}-\gamma\sqrt{\tau}\right)-\Psi\left(\frac{\alpha-\theta\tau}{\gamma\sqrt{\tau}}-\gamma\sqrt{t}\right)}{\Psi\left(\frac{\beta-\theta\tau}{\gamma\sqrt{\tau}}-\gamma\sqrt{\tau}\right)-\Psi\left(\frac{\alpha-\theta\tau}{\gamma\sqrt{\tau}}-\gamma\sqrt{\tau}\right)}\right] + Ke^{-r\tau}\left[\frac{\Psi\left(\frac{\ln\left(\frac{K}{S_0}\right)-\theta\tau}{\gamma\sqrt{\tau}}\right)-\Psi\left(\frac{\alpha-\theta\tau}{\gamma\sqrt{\tau}}\right)}{\Psi\left(\frac{\beta-\theta\tau}{\gamma\sqrt{\tau}}\right)-\Psi\left(\frac{\alpha-\theta\tau}{\gamma\sqrt{\tau}}\right)}\right]$$

Thus, the duality relation for the truncated normal distribution is;

$$\tilde{C}^{dual} = Ke^{r\tau}\left[\frac{\Psi(\nu)-\Psi(\varphi)}{\Psi(\lambda)-\Psi(\varphi)}\right] - S_0\left[\frac{\Psi(\nu-\gamma\sqrt{\tau})-\Psi(\varphi-\gamma\sqrt{\tau})}{\Psi(\lambda-\gamma\sqrt{\tau})-\Psi(\varphi-\gamma\sqrt{\tau})}\right] = P^{tnd} \tag{11}$$

Since equations (8) and (11) are equivalent, then the put-call duality relation hold for the TND model as it does with the B-S model in equation (7) with the assumption that −α = β and −β = α. This assumption is in conjunction with the symmetrical characteristic of the normal distribution in equation (10). Hence, proof.

Practical Applications: Calibration, Hedging, and Market Consistency

The inclusion of the put-call duality in the TND model provides not only practical advantages to investors and financial assets traders but also the theoretical benefits. In this section, we highlighted the application of the put-call duality in three domains: model calibration, delta hedging, and checking the model's consistency to the market with a focus on the duality relation to ensure robustness and the arbitrage inconsistencies reduction.

Model Calibration with Duality Constraint

Let C^{tnd} and P^{tnd} be the price of European call and put options under the truncated normal distribution model with current stock price of the underlying asset, S0, option strike price, K and time to maturity, τ as shown in equations (8) and (11) respectively. Therefore, under the risk-neutral valuation and assuming non-dividend paying stock, the put-call duality for the TND model correcting for the discounted strike price differential can be represented as:

$$C(\tau, S_0, K)^{TND} = P(\tau, S_0, K)^{TND} + (S_0 - K)\, e^{-r\tau} \tag{12}$$

The put-call duality identity presented in equation (12) holds under the assumption of symmetry as defined in equation (10) in the risk-neutral density with an appropriate change of measure as defined in equation (1).

Accurate calibration of pricing models to observed option market data is critical for effective valuation, hedging, and risk management. In the context of the Truncated Normal Distribution (TND), calibration involves estimating the model parameters ϑ = (θ,γ,α,β) where α and β represents the truncation bounds and θ and γ are the location and scale parameters. Traditional calibration minimizes pricing error against market-observed call or put prices. To enhance the model calibration and improve the robustness and consistency parameter estimation, we incorporated the put-call duality constraint into the model. The least-square calibration objective function can be formulated as;

$$\min_{\vartheta} \sum_{j=1}^{n} \left[\left(C^{\text{model}}(K_i;\vartheta) - C^{\text{mkt}}(K_j)\right)^2 + \kappa \left(C^{\text{model}}(K_j;\vartheta) - \tilde{P}^{\text{model}}(\tilde{K}_j;\vartheta) - \eta_j\right)^2 \right] \tag{13}$$

where:
$\eta_j = (S_0 - K_j)e^{-r\tau}$;
κ ≥ 0 = is a penalty parameter controlling the weight of the duality constraint
$\tilde{P}$model($\tilde{K}_j$;ϑ) = is the duality put price where the current stock price is exchanged with the strike price

The minimization of the calibration objective function in equation (13) enhances regularization of the model thereby reducing over-fitting and ensurers that the model put and call prices satisfies the symmetry in equation (10). Further, the duality constraint improves the stability of model parameters especially in sparse and noisy datasets, generalization of the model producing consistency results that align with theoretical principles and

Delta Hedging Strategies

Modern financial markets are often characterized by pricing instruments that are incomplete or asymmetric in liquidity. The put-call duality relation come in handy to help investors and traders in financial assets flexibility in hedging. This is due to the synthetic ability of the replication of call options using a put options. The application of the boundary condition on the asset prices under the TND model provides a more realistic and robust risk profile for European options unlike the BS model that assumes non-boundary assumption - specifically in portfolio hedging [31]. Under the TND model, the call option delta is given by;

$$\Delta^c = \frac{\partial V_c}{\partial S_0} = \left[\frac{\Phi\left[\lambda - \gamma\sqrt{\tau}\right] - \Phi\left[\nu - \gamma\sqrt{\tau}\right]}{\Phi\left[\lambda - \gamma\sqrt{\tau}\right] - \Phi\left[\varphi - \gamma\sqrt{\tau}\right]}\right] + \frac{1}{\gamma\sqrt{\tau}}\left[\frac{\phi\left[\nu - \gamma\sqrt{\tau}\right]}{\Phi\left[\lambda - \gamma\sqrt{\tau}\right] - \Phi\left[\varphi - \gamma\sqrt{\tau}\right]}\right]$$
$$-\frac{Ke^{-r\tau}}{S_0\gamma\sqrt{\tau}}\left[\frac{\phi[\nu]}{\Phi[\lambda] - \Phi[\varphi]}\right] \tag{14}$$

Considering the conditional filtration in equation (1), the call delta in equation (14) can be simplified as:

$$\Delta^c = \frac{\partial}{\partial S_0}\mathbb{E}^{\mathbb{Q}}\left[(S_t - K)^+\right] e^{-r\tau} \tag{15}$$

The boundaries imposed on the asset returns by the TND model ensures that the call delta reflects the price (payoff) of the option when there is minimal change of the current stock price, S0. The sensitivity of the derivative as shown in equation minimizes the option delta at the extreme option strike values. Additionally, since the put-call duality for the TND model holds, as shown in equation (11), the call delta of a European option can be inferred from the delta of the put option and vice-versa as shown in equation (16) below.

$$\Delta^c(S_0, K, r, \varsigma, \tau) = -\Delta^p(S_0, K, r, \varsigma, \tau) \quad \text{where} \quad \varsigma \quad \text{is the implied volatility} \tag{16}$$

The condition in equation (16) enables dynamic asset hedging by investors and financial assets traders particualry when assets are mispriced or when illiquidity is evident in the financial market. This delta hedging and duality minimizes losses that can be experienced by investors due to slight changes in the underlying asset prices and removing arbitrage opportunities through market corrections.

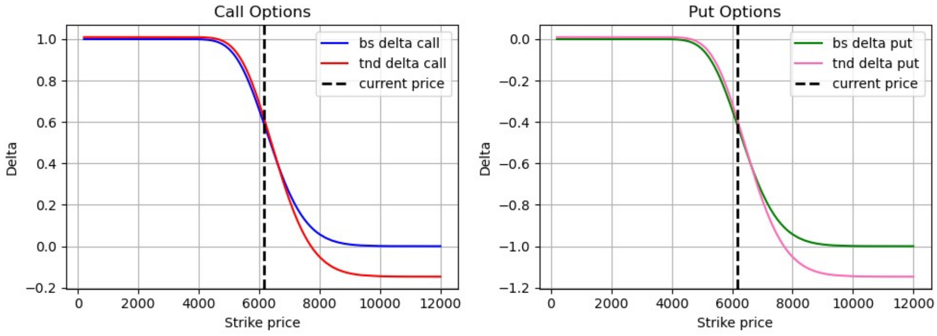

Figure 1: BS and TND Models Delta Plot

The put and call delta plots for the BS and TND models are presented in Figure (1). Clearly, the deltas are consistency for OTM options for both models. However, the TND is very sensitive for ITM options than the BS model.

Implementing delta hedging strategy using the TND model depicts significant improvement in hedging errors are shown in Figure (2). From the figure, the hedged portfolio under the BS model tends to follow closely the paths of the underlying asset prices. However, the portfolio value for the hedged portfolio under the TND model follows a distinct path than that of the underlying asset.

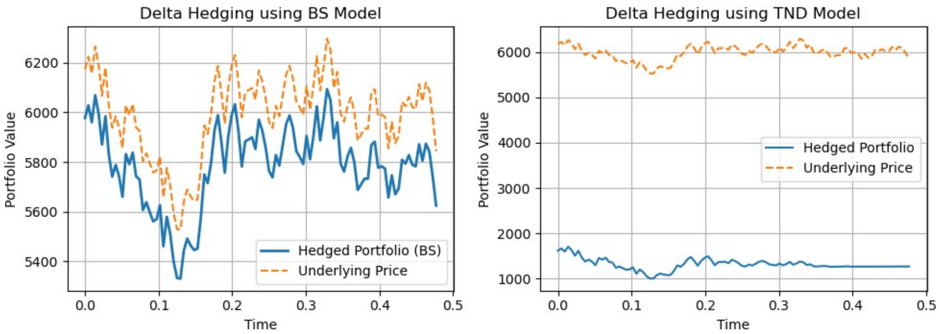

Figure 2: Delta Hedging: BS and TND Models

A plot of the hedged model errors relative to the final portfolio value is presented in Figure (3). It is evident that the errors for the BS and TND models are uniquely different though they tend to follow similar pattern. A comparison of the models in their delta hedging strategy is shown in Table (1). Clearly, TND model performs better than the BS model.

Model	n	Final Hedging Error	Mean Error	Std. Error
Black-Scholes	170	5624.3201	4593.0962	163.8371
Truncated Normal	103	1269.6380	120.7036	127.0853

Table 1: Delta Hedging Performance Comparison

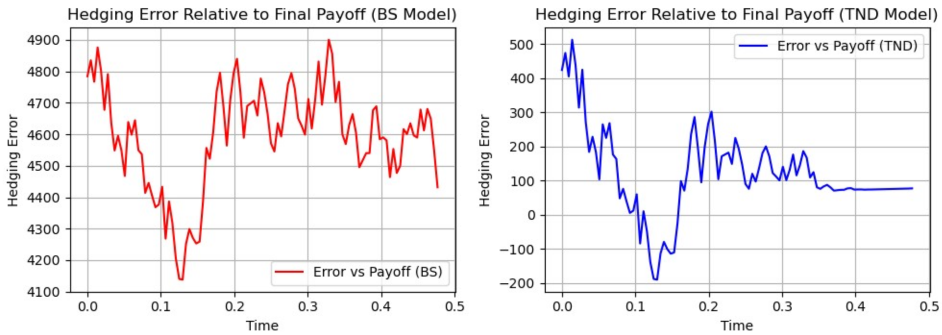

Figure 3: Delta Hedging Errors: BS and TND Models

Market Consistency Checks

The most commonly use of the put-call duality relationship in option pricing is to ensure that the model prices are consistent with observed market prices. The difference between the call and put prices should be captured by the corrected discounted differential of the strike price and current stock as shown in equation (13) and presented in equation (17)

$$C(\tau, S_0, K)^{TND} - P(\tau, S_0, K)^{TND} = (S_0 - K)\, e^{-r\tau} \tag{17}$$

Arbitrage opportunities or model mis-calibration will be captured by the TND model when there are pronounced deviation from equation (17). This is crucial when investors are engaged in product pricing that are structured, estimating of risk-neutral densities of their models or engaging real-time trading strategies where rigorous market consistency must be maintained. Market consistency duality checks is also crucial for industry regulators and compliance auditors to ensure that the distributions assumed by firms for price modeling does not encourage arbitrage opportunities. In this effect, the regulators and compliance may use a normal duality deviation index given by;

$$\pi(K) = \frac{\left|C^{mkt}(S_0, K, \tau) - P^{mkt}(K, S_0, \tau) - (S_0 - K)e^{-r\tau}\right|}{\bar{\Omega}} \tag{18}$$

where C^{mkt} and P^{mkt} are the market call and put option and $\bar{\Omega}$ is the average option price

Market consistency checks are critical to financial market regulators and assets investment auditors because it enables them to set a threshold π(K) that detects asset mispricing and the quality of model calibration. The duality check for the BS and TND model using the SPX index is presented in Figure (4). Clearly, the model pricing errors are minimal for the TND model than the BS model using the 25% violation threshold. Additionally, the errors profound at OTM for both models but only ITM for the TND model. Only few points that are outside the 0.25 threshold point under the proposed model. This indicates the consistency of the model in pricing.

Numerical Analysis and Results

In solving the duality relation, key model parameter are required. This include α,β,τ,γ,r, and θ. The truncation bounds are chosen by the investors depending on their risk appetite while the scale parameter is obtained from the historical stock price data of the underlying asset. However, the location parameter θ can be determined using any "root-finding" methodology appropriate to the individual by solving equation (19) which ensures that the martingale restriction proposed by Black and Scholes (1973), J. C. Cox and Ross (1976) and Merton (1973) is incorporated in the formulation of the TND Model to avoid the existence of arbitrage opportunities in the options trading market [15,31,33]. Finally, τ is the time to maturity

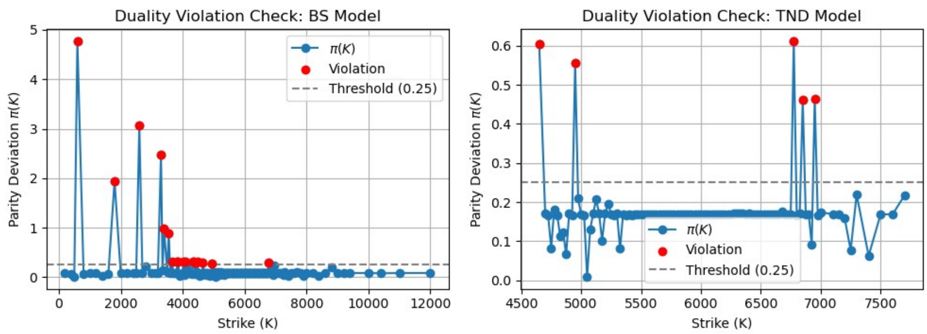

Figure 4: Duality Checks: BS and TND Models

(in years) of the European options. This study employed python's root-scalar optimization technique to estimate the location parameter, θ.

$$e^{\left(r-\theta-\frac{1}{2}\gamma^2\right)\tau} = \frac{\Psi\left(\frac{\beta-\theta\tau}{\gamma\sqrt{\tau}} - \gamma\sqrt{\tau}\right) - \Psi\left(\frac{\alpha-\theta\tau}{\gamma\sqrt{\tau}} - \gamma\sqrt{\tau}\right)}{\Psi\left(\frac{\beta-\theta\tau}{\gamma\sqrt{\tau}}\right) - \Psi\left(\frac{\alpha-\theta\tau}{\gamma\sqrt{\tau}}\right)} \tag{19}$$

The study employed secondary data for S&P 500 index (SPX) obtained from Yahoo Finance on June 27, 2025 that comprises of 1,255 five-year daily observations for a period of five years from June 29, 2020 to June 26, 2025 to estimate the location and scale parameters. The put and call option data with time to maturity of 174 days (0.477 years) maturing on December 19, 2025 were similarly retrieved from Yahoo Finance on June 27, 2027. The number of observations for call and put option were 171 and 184 respectively. The current stock price of the underlying asset captured on the same date was S0 = $6,141.02. The risk-free rate of interest, r = 4.25%, used in this study was the six-month Treasury Bill Rate obtained from the U.S. Department of the Treasury retrieved on June 26, 2025. This is because the time to maturity of the options is approximately six months as proposed by Shu and Zhang (2004) [34]. The truncation bounds is estimate as 25% from the current stock price implying that α and β is assumed to be ln0.75 and ln1.25 respectively. The number of observations for both the call and put options after truncation was 104. The estimated scale parameter, γ, from the historical data is 0.2098. Finally, the location parameter, θ obtained after solving equation (19) is approximately 0.077595. A plot for the convergence of the solution for the martingale restriction is shown in Figure (5).

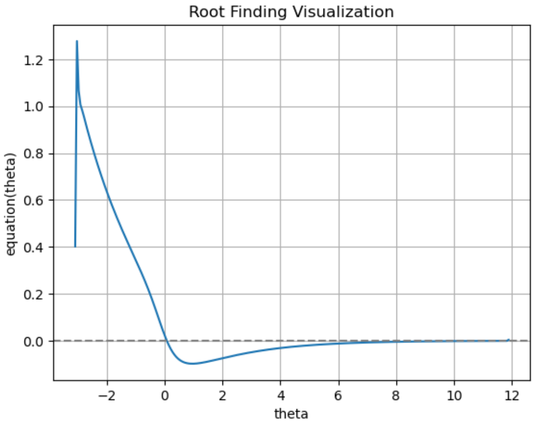

Figure 5: Martingale Restriction

Results and Discussion

Table (2) presents the first few estimated prices of S&P 500 index using the BS, TND and the duality models where K is the strike price, C^{mkt} and P^{mkt} are the observed market call and puts, C^{bs}, P^{bs} and $\tilde{C}^{bs}$, are the call, put and dual prices under BS model, and C^{tnd},P^{tnd} and $\tilde{C}^{tnd}$ are the call, put and dual prices under the TND model respectively.

K	Cmkt	Pmkt	Cbs	Pbs	C˜bs	Ctnd	Ptnd	C˜tnd
4650	1183.20	33.40	1622.045	5.005	5.005	1617.051	0.011	0.011
4700	1575.20	35.35	1574.206	6.155	6.155	1568.192	0.141	0.141
4725	1548.85	36.35	1550.365	6.809	6.809	1543.822	0.266	0.266
4750	1384.95	37.45	1526.581	7.520	7.520	1519.497	0.436	0.436
4775	1510.90	38.55	1502.858	8.291	8.291	1495.219	0.653	0.653
4800	1480.00	39.70	1479.199	9.128	9.128	1470.992	0.921	0.921
4825	1297.70	40.90	1455.610	10.033	10.033	1446.820	1.243	1.243
4850	1268.65	41.65	1432.094	11.012	11.012	1422.705	1.624	1.624
4875	1352.90	43.40	1408.655	12.068	12.068	1398.653	2.066	2.066
4900	1388.70	44.75	1385.298	13.206	13.206	1374.666	2.574	2.574
4925	1363.05	46.30	1362.027	14.430	14.430	1350.750	3.153	3.153
4950	930.95	47.55	1338.848	15.745	15.745	1326.908	3.805	3.805
4975	1104.45	49.00	1315.764	17.156	17.156	1303.144	4.537	4.537
5000	1295.25	50.55	1292.781	18.668	18.668	1279.464	5.351	5.351
5025	1271.55	52.20	1269.904	20.286	20.286	1255.872	6.254	6.254
5050	1149.30	53.80	1247.138	22.014	22.014	1232.373	7.249	7.249
5075	1058.00	55.50	1224.487	23.858	23.858	1208.971	8.343	8.343
5100	1205.40	57.30	1201.958	25.824	25.824	1185.672	9.539	9.539
5125	968.95	59.20	1179.555	27.916	27.916	1162.482	10.843	10.843
5150	1160.20	61.05	1157.284	30.139	30.139	1139.404	12.260	12.260
5175	983.10	63.05	1135.150	32.500	32.500	1116.445	13.795	13.795
5200	1115.25	65.10	1113.158	35.004	35.004	1093.610	15.455	15.455
5225	884.95	67.45	1091.315	37.655	37.655	1070.905	17.245	17.245
5250	1070.50	69.75	1069.625	40.460	40.460	1048.334	19.169	19.169
5275	1046.40	72.00	1048.094	43.423	43.423	1025.905	21.235	21.235
5300	1026.55	74.40	1026.727	46.551	46.551	1003.622	23.447	23.447
5325	862.05	76.80	1005.530	49.849	49.849	981.492	25.811	25.811
5350	982.00	79.30	984.508	53.322	53.322	959.519	28.333	28.333
5375	958.55	81.90	963.667	56.975	56.975	937.711	31.019	31.019
5400	938.30	84.60	943.011	60.814	60.814	916.072	33.875	33.875
5425	915.40	87.40	922.546	64.844	64.844	894.608	36.906	36.906

Table 2: Estimated Option Prices

Graphically, the values are presented in Figures (6), (7) and (8) for the model and market respectively.

Implied Volatility

Implied volatility (IV) is highly considered as a good measure of the market's expectation of the price fluctuation on an asset in the future. Although IV doesn't predict the direction of price movements, it provides an indication of the magnitude of expected price swings. High IV suggests a greater potential for price movement, while low IV suggests less movement is anticipated. Higher IV generally leads to higher option prices, as it reflects increased risk and uncertainty. Additionally, IV offers insights into market sentiment because high volatility depicts heightened risk perception, while low volatility suggest a more stable market. Therefore, implied volatility is considered a crucial metric for options traders, as it helps them assess the potential risk and reward of trading options and understand the market's expectations

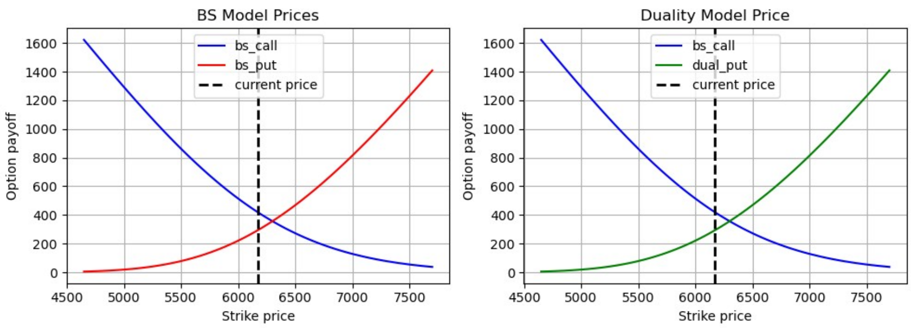

Figure 6: Option Pricing Using BS Model

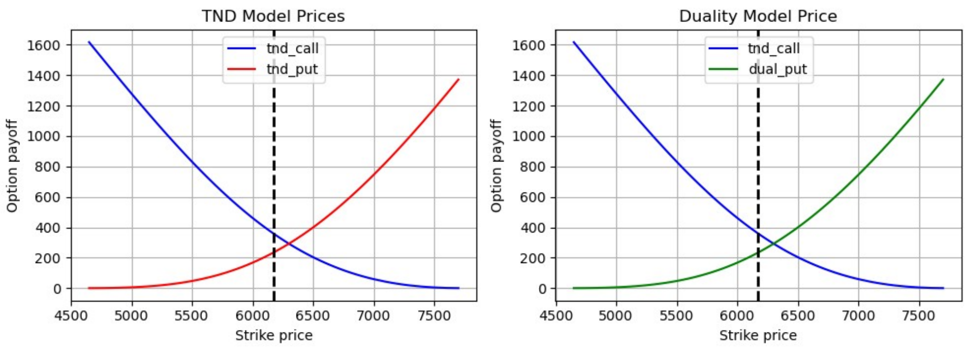

Figure 7: Option Pricing Using TND Model

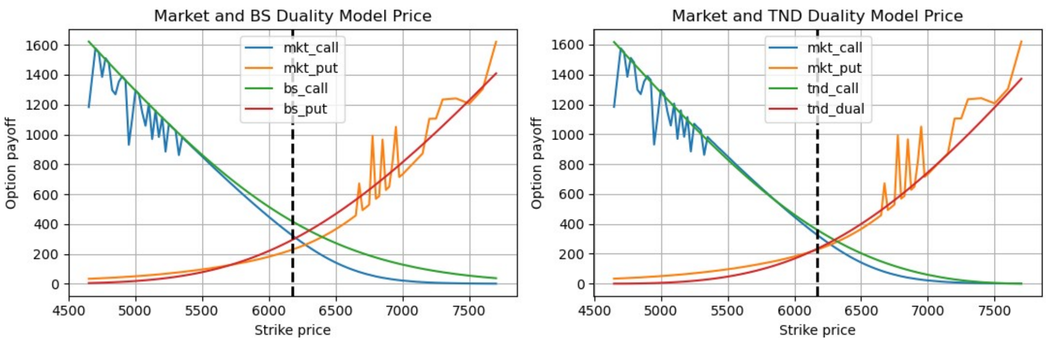

Figure 8: Market and Dual Models Prices

for future price movements. This study evaluated the volatility implied by the TND model in comparison with the famous Black & Scholes as presented in Table (3).

A plot of implied volatility against strike prices provided in Figures (9), (10), and (11).
The implied volatility for the BS, TND and observed market call options are represented by Figure (9).

It is evident that the market and BS model curve provides volatility smirk indicating that out-of-the-money (OTM) call options have higher implied volatility than at-the-money (ATM) call options. However, the curve of the implied volatility for the TND model is U-shaped (smile) though skewed signifying that the calls options are deep in-the-money and out-of-the-money options compared to at-the-money. Additionally, the TND models provided a closer estimated of the IV to the observed market IV than the BS model which seems to underestimate it.

The IV curves for put options are similarly presented in Figure (10). The market and BS model IV depicts a volatility smirk indicating that OTM put options have higher implied volatility than ATM put options. The truncated normal distribution model shows a volatility smile for put options as was the case with the call options. Critical to note is that the TND models over-predicted the IV for the ITM and OTM put options compared to the BS model.

strike	call	put	callvol	bscallvol	tndcallvol	putvol	bsputvol	tndputvol
5650	724.70	116.70	0.256691	0.190155	0.247785	0.192193	0.214835	0.858588
5675	704.00	120.45	0.253706	0.188636	0.242059	0.189793	0.212757	0.498735
5700	683.40	124.45	0.250706	0.187070	0.236726	0.187430	0.210774	0.418032
5725	662.95	128.40	0.247688	0.185495	0.231831	0.184952	0.208645	0.372881
5750	642.60	132.65	0.244652	0.183867	0.227174	0.182502	0.206639	0.344786
5775	622.40	136.85	0.241594	0.182224	0.222815	0.180003	0.204484	0.322868
5800	602.35	141.45	0.238640	0.180561	0.218697	0.177521	0.202515	0.307110
5825	582.50	145.30	0.235656	0.178912	0.214867	0.174980	0.199889	0.290355
5850	562.75	150.00	0.232636	0.177197	0.211099	0.172376	0.197767	0.279092
5875	543.10	154.85	0.229580	0.175414	0.207380	0.169772	0.195633	0.269300
5900	523.60	159.95	0.226545	0.173598	0.203774	0.167161	0.193550	0.260892
5925	504.35	165.10	0.223526	0.171814	0.200399	0.164469	0.191378	0.253038
5950	485.20	170.50	0.220458	0.169951	0.197012	0.161759	0.189247	0.246087
5975	466.35	176.10	0.217459	0.168142	0.193864	0.159019	0.187118	0.239742
6000	447.60	181.85	0.214400	0.166247	0.190668	0.156245	0.184954	0.233789
6025	429.15	187.85	0.211396	0.164393	0.187656	0.153426	0.182815	0.228333
6050	410.85	194.10	0.208324	0.162476	0.184627	0.150615	0.180694	0.223279
6075	392.80	200.55	0.205296	0.160555	0.181682	0.147740	0.178553	0.218477
6100	375.00	207.20	0.202307	0.158623	0.178800	0.144793	0.176388	0.213880
6125	357.50	214.25	0.199290	0.156706	0.176015	0.141881	0.174317	0.209728
6150	340.30	221.50	0.196356	0.154795	0.173305	0.138873	0.172208	0.205683
6175	323.40	229.10	0.193382	0.152882	0.170652	0.135816	0.170148	0.201912
6200	306.85	237.00	0.190477	0.150992	0.168086	0.132693	0.168098	0.198308
6225	290.70	245.30	0.187635	0.149146	0.165635	0.129548	0.166113	0.194959
6250	274.85	253.95	0.184788	0.147276	0.163188	0.126364	0.164155	0.191769
6275	259.85	263.00	0.181401	0.145699	0.161218	0.123120	0.162247	0.188769
6300	244.80	272.40	0.178641	0.143877	0.158911	0.119733	0.160353	0.185875
6325	230.20	282.30	0.175973	0.142098	0.156696	0.116298	0.158556	0.183227
6350	216.05	292.65	0.173331	0.140354	0.154560	0.112786	0.156818	0.180745
6375	202.40	303.85	0.170828	0.138668	0.152531	0.110305	0.155372	0.178823
6400	189.25	314.90	0.168339	0.137034	0.150597	0.105391	0.153623	0.176412
6425	176.55	327.10	0.165917	0.135417	0.148704	0.102703	0.152334	0.174831
6450	164.40	339.50	0.163562	0.133871	0.146928	0.098587	0.150942	0.173100

Table 3: Estimated and Observed Implied Volatility

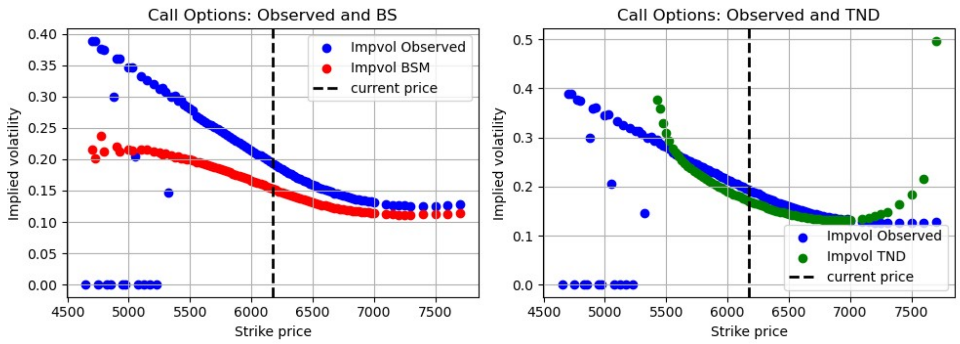

Figure 9: BS, TND and Observed Market Call Implied Volatility

Finally, we compared the implied volatility plots of the BS and TND models for both the call and puts options as shown in Figure (11). It is evident that the TND models reflects a volatility smile while the BS model depicted a volatility smirk for both the call and put options. Both models predicted closed IV for at-the-money options but large deviations for the OTM and ITM options. In general, the TND model predicted a higher future price fluctuations for the SPX options and hence requiring higher returns for the holders of the options. The higher IV for the TND model can also prompt traders to exercise their

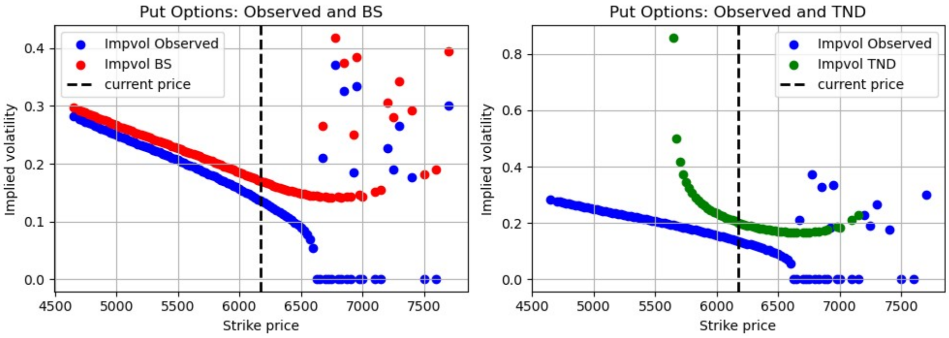

Figure 10: BS, TND and Observed Market Put Implied Volatility

rights to sell the options so that they can capitalize on potentially high premiums, while low IV may encourage them to buy options to benefit from anticipated price increases.

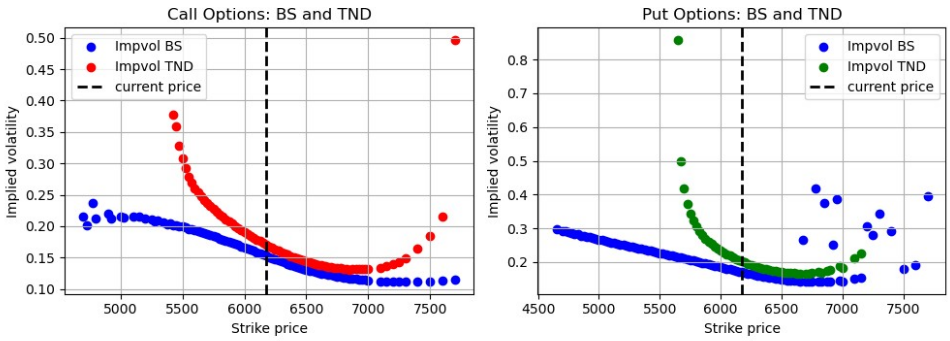

Figure 11: BS and TND Models Implied Volatility

Model Performance

The Akaike Information Criterion (AIC) and Bayesian Information Criterion (BIC) are information criteria used in model selection, including option pricing models. They balance model fit (likelihood) with model complexity (number of parameters) to help choose the most appropriate model. Whereas AIC favors models that fit the data well, even if they are slightly more complex, BIC tends to favor simpler models, especially with larger datasets, due to its stricter penalty for complexity [35]. The model that has the lowest performance metric should be considered as the appropriate model for option pricing. From Table (4), the TND has the lowest AIC and BIC for both call and put option and hence can be chosen as the best model for option pricing.

Model	Type	AIC	BIC	n
Truncated Normal	call	1182.80	1190.70	103
Black-Scholes	call	2652.94	2656.08	170
Truncated Normal	put	1195.03	1202.93	103
Black-Scholes	put	1941.52	1944.65	170

Table 4: AIC and BIC Index

Other metrics that are essential in evaluating the performance of a model are the Root Mean Squared Error (RMSE), Mean Absolute Error (MAE) and the Mean Relative Error (MRE). The result for the BS and TND model are presented in Table (5). The proposed model perform better the BS model for calls but inconsistent for put options.

Model	Type	RMSE	MAE	MRE (%)	n
Truncated Normal	call	82.3113	48.6381	41.3852	103
Black-Scholes	call	620.3128	200.9467	275.5499	170
Truncated Normal	put	86.5120	52.6900	36.3653	103
Black-Scholes	put	72.6703	47.0061	47.8192	170

Table 5: Pricing Errors

Conclusion, Recommendation and Future Work

This study introduces a novel framework for analyzing and enforcing put-call duality under the Truncated Normal Distribution (TND) model for European-style option pricing. Unlike the classical Black-Scholes model, which assumes log-normal returns and imposes put-call parity through its structure, the TND framework allows for bounded support and asymmetric distributions. To bridge this structural flexibility with theoretical consistency, we developed and validated a duality-penalized calibration method that integrates pricing symmetry into parameter estimation. The change of variable technique proved significant in the put-call duality constraint which merged the results obtained by the BS model. Theoretical derivations confirmed that a form of the duality identity can be constructed within the TND model under martingale conditions. We incorporated this identity into the calibration objective through a regularization term, which significantly improved pricing consistency and robustness. Our numerical experiments demonstrate that the duality-constrained TND model matches or exceeds Black-Scholes performance in fitting both synthetic and real-world option price data, particularly in regimes exhibiting skewness, truncation effects, or market noise. The Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), and the pricing error tests proved the outstanding performance of the proposed model. Additionally, enforcing duality improved market consistency checks, ensured smoother Greeks for hedging, and may served as a diagnostic tool to detect arbitrage violations or data anomalies. These practical benefits underscore the value of incorporating theoretical identities into empirical calibration routines. The implied volatility smiles also depicted the model's performance in price changes. Therefore, the study contributed both to the theoretical foundations and practical applications of symmetric pricing identities in non-Gaussian option pricing models. Promoting a structural consistency through duality calibration, the model becomes a more robust and interpretable tool for financial engineering and derivative analytics.

From the study findings, we recommend that financial institutions, particularly those engaged in pricing exotic derivatives or managing volatility surfaces, are encouraged to explore the TND-based pricing model as an alternative to the classical Black-Scholes framework. The duality-based calibration method offers enhanced pricing stability and arbitrage consistency, which are critical in fast-moving markets. Further, due to the model's consistency with put-call duality and ability to capture skewed distributions more accurately, we recommend risk managers to consider integrating the TND framework into Value-at-Risk (VaR) and scenario analysis engines. Its flexibility offers improved handling of tail risk and more accurate hedging under non-Gaussian market behavior. Additionally, we recommend that the regulators and oversight bodies such as the Basel Committee, and the European Securities and Markets Authority (ESMA) to consider including duality-consistent models in model validation frameworks. The dual-objective calibration criterion may serve as a diagnostic tool for identifying model mis-specification or market manipulation.

The future research work recommended from this study include investigating duality constraints in the context of entire volatility surfaces across strikes and maturities where a surface-based TND calibration will incorporate time-dependent truncation or stochastic bounds, extending the TND-based duality model to the pricing of American, barrier or digital options where early exercise or path dependency matters presents an open challenge. Additionally, an extension of duality principle to multi-asset options within truncated normal or elliptical copula frameworks could offer deeper insight into symmetry properties across assets is another area of future study. Finally, we recommend the incorporation of the TND-based duality constraint with machine learning models such as neural SDEs that could enable data-driven but theoretically grounded pricing models.

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References

1. Stringham, E. (2003). The extralegal development of securities trading in seventeenth-century Amsterdam. The Quarterly Review of Economics and Finance, 43(2), 321-344.
2. Bae, K. H., Jang, H., & Park, K. S. (2003). Traders' choice between limit and market orders: evidence from NYSE stocks. Journal of Financial Markets, 6(4), 517-538.
3. Alfonsi, A., & Jourdain, B. (2006). A call-put duality for perpetual American options. arXiv preprint math/0612648.
4. Alfonsi, A., & Jourdain, B. (2009). Exact volatility calibration based on a Dupire-type Call–Put duality for perpetual American options. Nonlinear Differential Equations and Applications NoDEA, 16(4), 523-554.
5. Carr, P., & Bowie, J. (1994). Static simplicity. Risk, 7(8), 45–50.
6. Carr, P., Geman, H., Madan, D. B., & Yor, M. (2002). The fine structure of asset returns: An empirical investigation. The journal of Business, 75(2), 305-332.
7. Chesney, M., & Jeanblanc, M. (2004). Pricing American currency options in an exponential Lévy model. Applied Mathematical Finance, 11(3), 207-225.
8. Swishchuk, A. (2023). Romancing with Black-Scholes Model: An Overview. WILMOTT. July, 126, 78-88.
9. Cox, M. A. A. (2009). Control charts for monitoring observations from a truncated normal distribution. The Journal

of Risk Finance, 10(3), 288-304.
10. del Castillo, J., & Daoudi, J. (2009). The mixture of left–right truncated normal distributions. Journal of statistical planning and inference, 139(10), 3543-3551.
11. Hasan, M. Z., Kamil, A. A., Mustafa, A., & Baten, M. A. (2012). Stochastic frontier model approach for measuring stock market efficiency with different distributions. PloS one, 7(5), e37047.
12. Norgaard, R., & Killeen, T. (1980). Expected utility and the truncated normal distribution. Management Science, 26(9), 901-909.
13. Onsoti, A. N. (2020). European Option Pricing Using Truncated Normal Distribution (Doctoral dissertation, University of Nairobi).
14. Zhu, S. P., & He, X. J. (2018). A modified Black–Scholes pricing formula for European options with bounded underlying prices. Computers & Mathematics with Applications, 75(5), 1635-1647.
15. Merton, R. (1973). Theory of rational option pricing. the bell journal of econometrics and management science, 4, 141-183.
16. Dumas, B., Fleming, J., & Whaley, R. E. (1998). Implied volatility functions: Empirical tests. The Journal of Finance, 53(6), 2059-2106.
17. Brechmann, E. C., Czado, C., & Aas, K. (2012). Truncated regular vines in high dimensions with application to financial data. Canadian journal of statistics, 40(1), 68-85.
18. Peiro, A. (1999). Skewness in financial returns. Journal of Banking & Finance, 23(6), 847-862.
19. Rachev, S. T., Menn, C., & Fabozzi, F. J. (2005). Fat-tailed and skewed asset return distributions: implications for risk management, portfolio selection, and option pricing. John Wiley & Sons.
20. Mwaniki, I. J. (2019). Geometric Brownian Motion Assumption and Generalized Hyperbolic Distribution on Modeling Returns. Journal of Advances in Applied Mathematics, 4(3).
21. Basnarkov, L., Stojkoski, V., Utkovski, Z., & Kocarev, L. (2019). Option pricing with heavy-tailed distributions of logarithmic returns. International Journal of Theoretical and Applied Finance, 22(07), 1950041.
22. Dey, O., & Chakraborty, D. (2012). A single period inventory model with a truncated normally distributed fuzzy random variable demand. International Journal of Systems Science, 43(3), 518-525.
23. Pender, J. (2015). The truncated normal distribution: Applications to queues with impatient customers. Operations Research Letters, 43(1), 40-45.
24. Hocquard, A., Papageorgiou, N., & Remillard, B. (2015). The payoff distribution model: an application to dynamic portfolio insurance. Quantitative Finance, 15(2), 299-312.
25. Gatti, D. D., Gallegati, M., Giulioni, G., & Palestrini, A. (2003). Financial fragility, patterns of firms' entry and exit and aggregate dynamics. Journal of economic behavior & organization, 51(1), 79-97.
26. Cao, Y., Liu, X., Zhao, Y., & Han, X. (2023). Pricing European Options Based on a Logarithmic Truncated t-Distribution. Journal of Applied Mathematics and Physics, 11(5), 1349-1358.
27. Fajardo, J., & Mordecki, E. (2006). Symmetry and duality in Lévy markets. Quantitative finance, 6(3), 219-227.
28. Eberlein, E., Papapantoleon, A., & Shiryaev, A. N. (2008). On the duality principle in option pricing: semimartingale setting. Finance and Stochastics, 12(2), 265-292.
29. Albrecher, H., Ladoucette, S. A., & Schoutens, W. (2007). A generic one-factor Lévy model for pricing synthetic CDOs. In Advances in mathematical finance (pp. 259-277). Boston, MA: Birkhäuser Boston.
30. Taleb, N. N. (2014). Risk neutral option pricing with neither dynamic hedging nor complete markets. arXiv preprint arXiv:1405.2609.
31. Black, F., & Scholes, M. (1973). The pricing of options and corporate liabilities. Journal of political economy, 81(3), 637-654.
32. Peskir, G., & Shiryaev, A. N. (2002). A note on the call-put parity and a call-put duality. Theory of Probability & its Applications, 46(1), 167-170.
33. Cox, J. C., & Ross, S. A. (1976). The valuation of options for alternative stochastic processes. Journal of financial economics, 3(1-2), 145-166.
34. Shu, J., & Zhang, J. E. (2004). Pricing S&P 500 index options under stochastic volatility with the indirect inference method. Journal of derivatives accounting, 1(02), 171-186.
35. Montag, M., & Persson, F. (2015). Model risk quantification in option pricing.