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Quantization of Hyper-Complex Gauges and Space Time: Dering the Fine Structure Constant as a Dimensionless Geometric Constant Beyond 12th Digits

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Abstract
To derive the precise fine structure constant, we use an approach based on hypercomplex algebra, including Hamilton's 4D quaternions, Cayley's 8D octonions, and 16D sedenions, which have broad applications in particle physics. This framework allows us to investigate electron quantum dynamics, introducing a hypercomplex non-Abelian, non-associative gauge for the Dirac equation with internal structure. Extending 4D spacetime to a higher-dimensional lattice, we show that electron coupling to a quantized gauge field yields an effective quantized mass. This approach derives an inverse fine-structure constant of 137 from the su (2) octonion gauge and a precise experimental match of 137.035999206 from the $(\mathrm{su}(2) \oplus \mathrm{su}(2) \oplus \mathrm{su}(2)) \times S_3 \oplus \mathrm{su}(2)$ sedenion gauge. These gauges dictate lepton masses in higher-dimensional generalized Einstein's massenergy relation. The fundamental constant, linked to Pythagorean primes, governs electron-photon interactions and plays a crucial role across physics.

Keywords: Fine Structure Constant, Einstein's Mass-Energy Relation, Dirac Equation, Hyper complex Gauge Theory, Octonion, Sedenion and Pythagorean Prime

Introduction
The fine-structure constant is an important physical parameter that dictates the coupling strength of the electromagnetic interactions between charged particles or between charged or magnetic particles in an external electromagnetic field [1]. Its value has been determined with unprecedented accuracy to the 10th decimal digit [2,3]. However, its origin for why this constant takes its value has been a century-old mystery. Many great physicists, including Pauli, Dirac, Feynman, Weinberg, and many others have been puzzled by this dimensionless constant and have commented on its baffling origin. In this work, we propose an approach based on hypercomplex algebra, generalized gauge, Dirac equation, and spacetime quantization to answer the longstanding question. The imaginary number, invented by Cardano in the 16th century to solve general polynomial equations, its extension to a complex-value system has become the foundation of mathematics [4]. The complex numbers are also essential in physics, and are indispensable in describing Newtonian dynamics, Maxwell's equation of electromagnetism, special and general relativity, quantum mechanics, etc. Since Hamilton invented quaternions in the mid-18th century, higher dimensional hyper complex algebra has found many applications, which can be constructed using the Cayley-Dickson scheme, layer by layer from lowest-dimensional real number system to 2D complex numbers, then to 4D quaternion, 8D octonion, and 16D sedenion algebra [5-9]. Recently, with the advances in computer technologies, quaternions have been used in computer graphics and gaming technologies, and software development for aviation and flight simulation. In physics, qualifications could be applied to special relativity and quantum theory. Other than quaternions, higher-dimensional hyper complex algebra, such as octonions and sedenions, has found applications in particle physics as a foundation for theoretical development beyond the Standard Model [7, 10]. In this work we shall first introduce what hypercomplex numbers are, the applications to electromagnetism, special relativity, Dirac equation, U (1) Lorentz gauge symmetry, hyper complex gauge, mass

acquisition, quantized gauge and mass generalized Dirac equation, and finally the derivation of the fine structure constant [11,12].

Hypercomplex Algebra and Applications

Number theory is not only an important subject in mathematics but also an essential foundation for the description of a physical system involving Newtonian dynamics, Maxwell's electromagnetism, Einstein's special and general relativity, Schrödinger's wave mechanic, Heisenberg's matrix mechanism in non-relativistic quantum theory, relativistic electrodynamics and (QED) and quantum chromodynamics (QCD) [13,14]. Complex numbers are indispensable for describing the periodic phenomena in classical and quantum systems, even though all physical measurements only involve real numbers. The use of the complex number system is necessary in many branches of physics, for example, the Minkowski space in relativity involves an imaginary time, Schoedinger's equation involves complex-valued wave amplitude, and Heisenberg matrix mechanics involves time and momentum operators with an imaginary derivative in time and space. All periodical dynamics, such as pendulum motion, orbiting satellites and planets, sound waves, electromagnetic waves, and AC circuitry, with a frequency can be described by Euler's famous identity relation of $e^{i\omega t}= \boldsymbol{cos}(\omega t) + i\boldsymbol{sin}(\omega t)$, which is the foundation of the Fourier analysis.

In this work, we will explain the concepts of hypercomplex numbers and various applications in physics, especially quantum theory for the electron, Dirac's equation, gauge fields, and generalized Dirac equation based on octonions and sedenions. Then, we will show how to derive the fine-structure constant, based on the lattice spacetime with quantized gauge and electron's mass in terms of the octonion and sedenion operators.

Construction of high-dimensional hyper complex algebra

Here we discuss the mathematical structures of hypercomplex number systems. They are extended, via the Cayley-Dickson construction scheme layer upon layer, from 2D complex algebra to a higher dimensional algebra [6].

Complex	$\mathbb{C} = (\mathbb{R}_1, \mathbb{R}_2)$	$(\mathbb{R}_1, \mathbb{R}_2)(\mathbb{R}_3, \mathbb{R}_4) = (\mathbb{R}_1\mathbb{R}_3 - \mathbb{R}_4\mathbb{R}_2 , \mathbb{R}_4\mathbb{R}_1 + \mathbb{R}_2\mathbb{R}_3)$
Quaternion	$\mathbb{Q} = (\mathbb{C}_1, \mathbb{C}_2)$	$(\mathbb{C}_1, \mathbb{C}_2)(\mathbb{C}_3, \mathbb{C}_4) = (\mathbb{C}_1\mathbb{C}_3 - \mathbb{C}_4^*\mathbb{C}_2 , \mathbb{C}_4\mathbb{C}_1 + \mathbb{C}_2\mathbb{C}_3^*)$
Octonion	$\mathbb{O} = (\mathbb{Q}_1, \mathbb{Q}_2)$	$(\mathbb{Q}_1, \mathbb{Q}_2)(\mathbb{Q}_3, \mathbb{Q}_4) = (\mathbb{Q}_1\mathbb{Q}_3 - \mathbb{Q}_4^*\mathbb{Q}_2 , \mathbb{Q}_4\mathbb{Q} + \mathbb{Q}_2\mathbb{Q}_3^*)$
Sedenion	$\mathbb{S} = (\mathbb{O}_1, \mathbb{O}_2)$	$(\mathbb{O}_1, \mathbb{O}_2)(\mathbb{O}_3, \mathbb{O}_4) = (\mathbb{O}_1\mathbb{O}_3 - \mathbb{O}_4^*\mathbb{O}_2 , \mathbb{O}_4\mathbb{O} + \mathbb{O}_2\mathbb{O}_3^*)$

Table 1: Cayley-Dickson's Hyper Complex-Number Construction Scheme

According to Table 1, a complex number can be constructed by a pair of real numbers with the prescribed multiplication rule. Because there is an isomorphism between a complex plane and 2D plane of paired real numbers, one can essentially use only a real number system to describe any periodical dynamics in a classical and non-relativistic quantum system, and except that the equation's explicit expression using the complex system is much simpler and concise. Such an equivalent description between the linear algebra of complex plane and 2D real-number plane no longer holds in relativistic quantum theory involving anti-commutative matrices unless one extends the complex algebra to complex non-commutative Dirac gamma matrices [15]. In octonion and sedenion algebra, the operators are not only anti-commutative but also non-associative; therefore, they could be represented by associative matrices. As shown in Fig. 1, the structure of sedenios consists of four quartets a 4D quaternion $\{\boldsymbol{I}, \boldsymbol{\Gamma}_1, \boldsymbol{\Gamma}_2, \boldsymbol{\Gamma}_3\}$, and three q uartets such as $\{\boldsymbol{\Theta}_1, \boldsymbol{U}_1, \boldsymbol{U}_2, \boldsymbol{U}_3\}$ which is part of octonion, and two other quartets, $\{\boldsymbol{\Theta}_2, \boldsymbol{V}_1, \boldsymbol{V}_2, \boldsymbol{V}_3\}$ and $\{\boldsymbol{\Theta}_3, \boldsymbol{W}_1, \boldsymbol{W}_2, \boldsymbol{W}_3\}$. Unlike the unity element I, whichforms a cyclic spinor triplet, they all anticommutative among themselves, and with the other spinor triplet sets, such as $\{\boldsymbol{U}_1, \boldsymbol{U}_2, \boldsymbol{U}_3\}$, ⊛$V_1$, V_2, V_3 $\{\boldsymbol{W}_1, \boldsymbol{W}_2, \boldsymbol{W}_3\}$ and $\{\boldsymbol{I}, \boldsymbol{\Gamma}_1, \boldsymbol{\Gamma}_2, \boldsymbol{\Gamma}_3\}$.

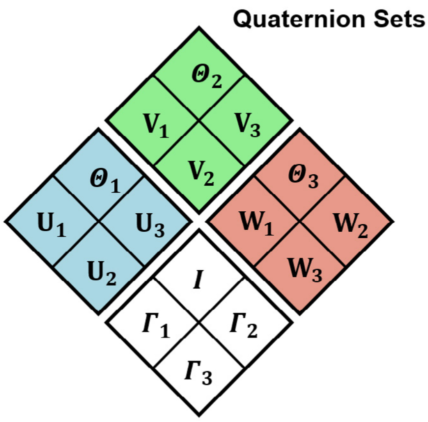

Figure 1: The Schematic Diagram Showing the Mathematical Structure of 16D Sedenion Algebra Consists Of One Quaternion Set and Three Other Quartets Which Contain Their Pseudo Scalar Operator $\boldsymbol{\Theta}_k$, Which Anti-Commutates With All Other Spinor Triplet Members

I and represents the exterior space-time, $\boldsymbol{\Theta}_k$ represents the pseudo-temporal operator, together with U_k, V_k, and W_k spatial operators form three quartets that play an important role for three generations of leptons and quarks. Altogether, these twelve anti-commutative operators describe the internal 12D spacetime.

In the mid-18th century Hamilton invented quaternion algebra, which possesses four degrees of freedom, represented by four basic elements, $\{1, \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$ according to Hamilton's original notation, or $\{\boldsymbol{I}, \boldsymbol{\Gamma}_1, \boldsymbol{\Gamma}_2, \boldsymbol{\Gamma}_3\}$in our notation to avoid confusion with integer indices with these basis quaternion elements. The multiplication tale of quaternions is given in Table 2 [7].

There are basis elements of imaginary numbers with $\boldsymbol{\Gamma}^2_k = -\boldsymbol{I}, \{\boldsymbol{\Gamma}_i, \boldsymbol{\Gamma}_j\} = 0, if\, i \neq j$.

$\boldsymbol{I}$	$\boldsymbol{\Gamma}_1$	$\boldsymbol{\Gamma}_2$	$\boldsymbol{\Gamma}_3$
$\boldsymbol{\Gamma}_1$	$-\boldsymbol{I}$	$\boldsymbol{\Gamma}_3$	$-\boldsymbol{\Gamma}_2$
$\boldsymbol{\Gamma}_2$	$-\boldsymbol{\Gamma}_3$	$-\boldsymbol{I}$	$\boldsymbol{\Gamma}_1$
$\boldsymbol{\Gamma}_3$	$\boldsymbol{\Gamma}_2$	$-\boldsymbol{\Gamma}_1$	$-\boldsymbol{I}$

Table 2: The Multiplication Table of Quaternion's $\{I, \Gamma_1, \Gamma_2, \Gamma_3\}$ With One Unity Element and Three Anti-Commutative Imaginary Basis Elements

Quaternions can be used to describe Minkowski space and Maxwell equation. Three anticommutative operators, SU (2) spinor triplet, can be represented by Pauli matrices. They can relate to 3D rotations, thus have been applied in gaming and aviation software technology. One can use Pauli's three 2 by 2 anti-commutative matrices $\sigma_k$to represent these quaternion basis elements by defining$\boldsymbol{\Gamma}_k = i\sigma_k$, where $\sigma_i\sigma_j = {}_{ijk}\sigma_k + \delta_{ij}\boldsymbol{I}$. When Dirac attempted to develop a relativistic wave equation for the electron because three Pauli matrices are not sufficient, he invented four anticommutative operators which represented by the tensor product of three Pauli matrices and an identity matrix. However, for a massless fermion, three SU (2) spinor operators in quaternions are sufficient. One could use quaternions to describe the electric and magnetic fields. Defining$\boldsymbol{E} = E_k\boldsymbol{\Gamma}_k, \boldsymbol{B} = iB_k\boldsymbol{\Gamma}_k$, a quaternion field$F = (\varphi, \boldsymbol{E} + i\boldsymbol{B})$,and a derivative quaternion operator $D = (\partial t, \nabla)$, one can express Maxwell equations, $\nabla \cdot \boldsymbol{E} = \rho, \nabla \cdot \boldsymbol{B} = 0, \nabla \times \boldsymbol{E} = -\partial \boldsymbol{B}/\partial t, \nabla \times \boldsymbol{B} = \boldsymbol{J} + \partial \boldsymbol{E}/\partial t$, by a single quaternion form as $DF = (\rho, \boldsymbol{J})$, and the Maxwell field tensor by F = D A− A D, where ρ is the electric density, and $\boldsymbol{J}$the current density.

Years after Hamilton's inventions of 4D quaternions, Cayley extended the work to octonions, which contains seven anti-commutative imaginary operators and an identity basis element. With their multiplication table given in Table 3 [8].

I	**Γ1**	**Γ2**	**Γ3**	**Θ**	**U1**	**U2**	**U3**
Γ1	**-I**	**Γ3**	**-Γ2**	**U1**	**-Θ**	**- U3**	**U2**
Γ2	**-Γ3**	**-I**	**Γ1**	**U2**	**U3**	**-Θ**	**- U1**
Γ3	**Γ2**	**-Γ1**	**-I**	**U3**	**- U2**	**U1**	**-Θ**
Θ	**-U1**	**- U2**	**-U3**	**-I**	**Γ1**	**Γ2**	**Γ3**
U1	**Θ**	**- U3**	**U2**	**–Γ1**	**-I**	**-Γ3**	**Γ2**
U2	**U3**	**Θ**	**- U1**	**–Γ2**	**Γ3**	**-I**	**-Γ1**
U3	**- U2**	**U1**	**Θ**	**–Γ3**	**-Γ2**	**Γ1**	**-I**

Table 3: The Multiplication Table of Octonions Consist Of One Identity Element and Seven Anticommutative Basis Elements

In the octonion algebra, the quaternion set of $\{\boldsymbol{I}, \boldsymbol{\Gamma}_1, \boldsymbol{\Gamma}_2, \boldsymbol{\Gamma}_3\}$, represents the exterior Minkowski spacetime, and another 4-element set $\{\boldsymbol{\Theta}. \boldsymbol{U}_1, \boldsymbol{U}_2, \boldsymbol{U}_3\}$, represents the internal spacetime of a particle. The operator $\boldsymbol{\Theta}$ is a pseudo time operator that causes conversion between two SU (2) spinors sets of $\boldsymbol{\Gamma}k$, and $\boldsymbol{U}_k$. We shall use octonion algebra as an alternate to generalize the Dirac equation for the electron.

Dirac equation and Coulomb scalar gauge

To derive a quantum wave equation for a relativistic electron, which is comparable with special relativity with the space and time variables equal footing, Dirac proposed a coupled linear differential equation using four anti-commutative matrices. His equation can be expressed in the covariant form involving four anti0commutative gamma matrices as

$$(i\gamma^{\mu}\partial_{\mu}-m)\Psi=0,\mu=0,1,2,3$$

$$p_k=-i\partial_k,p_0=i\partial_0$$
$$\gamma^k=\begin{pmatrix}0 & \sigma_k\\ -\sigma_k & 0\end{pmatrix},\gamma^0=\begin{pmatrix}\boldsymbol{I}_2 & 0\\ 0 & -\boldsymbol{I}_2\end{pmatrix},(\gamma^0)^2=-(\gamma^k)^2=\boldsymbol{I}_{,4}\,k=1,2,3$$
$$\sigma_1=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\sigma_2=\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix},\sigma_3=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},\{\sigma_j,\sigma_k\}=2\delta_{jk}\boldsymbol{I}_2, \qquad (1A)$$

σ_1 = (1 0) `, $\sigma_2$ = ($i$ 0), $\sigma_3$ = (0 −1) `, $\{\sigma_j, \sigma k\} = 2\delta_{jk}\boldsymbol{I}_2$, where the natural unit of $\hbar = c = 1$ is used. Dirac introduced the use of four anti-commutative operators so that the square of the equation would lead to Einstein's mass-energy relation of $E^2 = p1^2 + p2^2 + p3^2 + m0^2$. In the presence of an electric or magnetic field, one has

$$\left(i\gamma^{\mu}(\partial_{\mu}-eA_{\mu})\ \ -m\right)\Psi=0,$$
$$\partial_{\nu}F^{\mu\nu}=\partial^{\mu}A^{\nu}-\partial^{\nu}A^{\mu}=J^{\mu\nu}. \qquad (1B)$$

Where $J^{\mu\nu} = \Psi^{+}\gamma^{\mu}\Psi$, and the momentum operator needs to be replaced by$p_{\mu} \to p_{\mu} - eA_{\mu}$, A_{μ} is the electromagnetic four-potential, and $F\mu\nu$ is the Maxwell field tensor. The above equation is the foundation for quantum electrodynamics. One can relate the four-potential A_{μ} to the electric and magnetic fields by $\boldsymbol{E} = -\nabla\varphi - \partial\boldsymbol{A}/\partial t, \boldsymbol{B} = \nabla \times \boldsymbol{A}.$ Eq. (1) is invariant under local U (1) gauge transformation of the gauge function $\lambda\,(t, \boldsymbol{r})$ that satisfies

$$A\rightarrow A-\nabla\lambda(t,\boldsymbol{r}),\phi\rightarrow\phi+\frac{\partial}{c\partial t}\lambda(t,\boldsymbol{r})$$
$$\frac{\partial^2}{c^2\partial t^2}\lambda(t,\boldsymbol{r})-\nabla^2\lambda(t,\boldsymbol{r})=0. \qquad (2A)$$

We define dimensionless four-potential $\Lambda\,(t, \boldsymbol{r})$ and dimensionless cou0ling strength $\sqrt{\hbar c/e^2}$ for $\lambda(t,r)$ which is related to electric potential ϕ and vector potential A

$$\lambda(t,\boldsymbol{r})\rightarrow\sqrt{\frac{\hbar c}{e^2}}\Lambda(t,\boldsymbol{r})$$
$$e^{i\lambda(t,\boldsymbol{r})}\Psi(t,\boldsymbol{r})$$
$$\rightarrow e^{i\sqrt{\hbar c/e^2}\Lambda(t,\boldsymbol{r})}\Psi(t,\boldsymbol{r}) \qquad (2B)$$

The above scalar gauge with local transformation represents U (1) symmetry with one degree of freedom. The U (1) plays an important role in QED and the Standard Model that is based on the U (1) × SU (2) × SU (3) symmetry. We shall generalize this gauge function to a hypercomplex gauge using higher dimensional octonions and sedenions.

Generalized Dirac Equation and Hyper complex Gauge Based on Octonion Algebra

In QED, the Lorentz gauge and Dirac equation assume a point-like electron, which is invariant under gauge transformation [12]. However, if the electron is not a point-like particle but has a finite size, the U (1) gauge symmetry would be broken. As an alternative to Dirac's theory, this generalized Dirac equation with octonions assumes a finite size and an internal structure for the electron. Therefore, the electron could acquire a rest mass. To describe the quantum dynamics of a particle in the physical world, in addition to the length and time unit in 4D spacetime, a particle's mass or energy is required for a full physical description. That is why in Einstein's massenergy relation $E^2 = c^2(P1^2 + P1^2 + P2^2 + P3^2 + m0^2c^2)$ for a relativistic particle, the mass plays like an extra dimension in this Pythagorean formula for a 4D orthogonal structure consisting of three dimensions in momentum space and an extra dimension in mass. Therefore, for a quantized spacetime lattice, mass should be treated as an operator as the 5th dimension, in addition to the 4D spacetime.

In the quantized space-time lattice, the continuous differential wave equation becomes a discrete difference equation involving an equation involving integers in terms of the fundamental lattice units. With the quantized spacetime, our generalized gauge represents a gauge for the symmetry-broken Lorentz gauge so that the bare electron can acquire a rest mass. Physically speaking, those four octonion operators $\{\boldsymbol{\Theta}, \boldsymbol{U}_1, \boldsymbol{U}_2, \boldsymbol{U}_3\}$ do not commute with the quaternion operators $\{\boldsymbol{\Gamma}_1, \boldsymbol{\Gamma}_2, \boldsymbol{\Gamma}_3\}$ which describe the Lorentz invariant Minkowski 4D spacetime. The spinor set of the octonion algebra $\{\boldsymbol{U}_1, \boldsymbol{U}_2, \boldsymbol{U}_3\}$ is obtained by multiplying the quaternion spin set $\{\boldsymbol{\Gamma}_1, \boldsymbol{\Gamma}_2, \boldsymbol{\Gamma}_3\}$ by $\boldsymbol{\Theta}$, where $\boldsymbol{\Theta}$ is a pseudo-temporal operator that dictates the intrinsic rest mass of the particle. The complexified operator $i\boldsymbol{\Theta}$ dictates the decay because it becomes non0Hermitian and breaks the T and CP symmetries in wick interaction.

By quantization of the gauge $\Lambda\,(t, \boldsymbol{r})$ or e² c (t, r), which leads to the discrete value of the rest mass through $e^{i\sqrt{\hbar c/e^2}\Lambda(t,\boldsymbol{r})}$ must be a periodic function of the fundamental frequency related to the rest mass, and$\sqrt{\hbar c/e^2}\,\Lambda$ in the Fourier domainmust be an integer multiple of the fundamental wave vector or momentum Pkwhich is coupled to $\boldsymbol{U}k$. Therefore, for the most fundamental frequency mode in terms of the electron's mass, we obtainthe following equation involving integers

$$i\sqrt{e^2/\hbar c}\,(a_1\mathbf{U}_1 + a_2\mathbf{U}_2 + na_3\mathbf{U}_3 + a_0\mathbf{\Theta}) = \mathbf{I}. \qquad (3A)$$

Because $\boldsymbol{U}2k = -\boldsymbol{I}$ and $\{\boldsymbol{U}j, \boldsymbol{U}k\} = -2\delta ij$, the square of the above equation becomes

$$a_1^2 + a_2^2 + a_3^2 + a_3^2 = \hbar c/e^2 . \qquad (3B)$$

Where c / e^2 must be a prime number for the fundamental model and not a harmonic of the the most fundament mode. According to our previous work, we have shown this approach of quantizing the Lorentz gauge, we can derive the fine structure constant e^2 c=1137 [16].

As mentioned earlier, unlike the QED that assumes the gauge is a scalar function with U (1) symmetry, in our model with a gauge represented by $\{\boldsymbol{\Theta}, \boldsymbol{U}_1, \boldsymbol{U}_2, \boldsymbol{U}_3\}$ of the octonion algebra. The symmetry-breaking for the octonion gauge leads an electron to acquire an effective rest mass. Because quantization of the gauge is equivalent to quantizing the effective mass, we could directly deal with the Hamiltonian H. We define $\mathbf{H}|\boldsymbol{\Psi}\rangle \equiv -iE|\boldsymbol{\Psi}\rangle$ and for a relativistic electron with H=$_k$ P$_k$U$_k$ +Θm$_0$ in the natural unit, one obtains $E^2|\boldsymbol{\Psi}\rangle = -\mathbf{H}_2|\boldsymbol{\Psi}\rangle$, and E^2 = k Pk^2 + m0^2 which is Einstein's mass-energy relation.According to this octonion model, the generalized Dirac equation contains two parts, an exterior and an internal spacetime for the gauge that governs the internal dynamics for an electron to acquire an effective rest mass. We consider

$$\begin{aligned}
&\mathbf{H} = \textstyle\sum_k p_k \mathbf{\Gamma}_k + \mathbf{M} \\
&\mathbf{M} = \mathbf{P} + \mathbf{\Theta}\,\Omega_0 \\
&\mathbf{P} = \textstyle\sum_k P_k \mathbf{U}_k \\
&\{\mathbf{\Gamma}_i, \mathbf{\Gamma}_j\} = \{\mathbf{U}_i, \mathbf{U}_j\} = -2\delta_{ij}\mathbf{I}.\ \{\mathbf{\Gamma}_i, \mathbf{U}_j\} = \{\mathbf{\Gamma}_i, \mathbf{M}\} = \{\mathbf{M}, \mathbf{U}_j\} = 0.
\end{aligned} \qquad (4A)$$

With quantized mass and internal spacetime and the rest mass representing the fundamental mode, one obtains the following constraint equations for the relevant integers

$$\begin{aligned}
&a_1^2 + a_2^2 + a_3^2 + a_0^2 = \hbar c/e^2 \\
&a_4^2 = a_1^2 + a_2^2 + a_3^2
\end{aligned} \qquad (4B)$$

As shown in Table 3, based on the above equation of the constraints, we obtained
c/ e^2 =137, a_0 =4, a_4 =11, a_1=2, a_2 =6, a_3=9, leading to a theoretical value for the fine structure constant of 1/137.

$\hbar c/e^2$	$\mathbf{P} = P_1\mathbf{U}_1 + P_2\mathbf{U}2 + P_3\mathbf{U}_3$	$\Omega_0\mathbf{\Theta}$	$a_0, a_4: prime$ $a_1, a_2, a_3, a_0: integers > 1$
$\hbar c/e^2 = {a_4^2}^2 + a_0^2$ $a_0, a_4: prime$	$Triplet: \{a_1, a_2, a_3\}$ $a_4^2 = a_1^2 + a_2^2 + a_3^2$	a_0^2	$\lvert mod(R/a_0, \pi)\rvert < 10^{-2}$ $R = \prod_{k=0}^{3} a_k,\ \ a_0 = \sum_{k=0}^{3} a_k^2$

Table 4: Constraints on Integer Equations According to the Octonion Lattice Model

Generalized Dirac Equation and Hyper complex Gauge Based On Sedenion Algebra

To improve the theoretical prediction of the fine-structure constant to match the experimental value with am unprurient high accuracy, we extend the 8D octonion model for the gauge or the rest mass to another level up of the hyper-complex algebra, i.e., the 16D sedenion model. Using the Cayley-Dickson construction scheme as shown in Table 3, one can construct the 16-element sedenions from octonions. There are fifteen anti-commutative and non-associative imaginary basis elements, and their multiplication table is shown in Table 5 [9].

I	Γ$_1$	Γ$_2$	Γ$_3$	Θ$_1$	U$_1$	U$_2$	U$_3$	Θ$_2$	V$_1$	V$_2$	V$_3$	Θ$_3$	W$_1$	W$_2$	W$_3$
Γ$_1$	-I	Γ$_3$	-Γ$_2$	U$_1$	-Θ$_1$	- U$_3$	U$_2$	V$_1$	-Θ$_2$	- V$_3$	V$_2$	-W$_1$	Θ$_3$	W$_3$	-W$_2$
Γ$_2$	-Γ$_3$	-I	Γ$_1$	U$_2$	U$_3$	-Θ$_1$	- U$_1$	V$_2$	V$_3$	-Θ$_2$	- V$_1$	-W$_2$	-W$_3$	Θ$_3$	W$_1$
Γ$_3$	Γ$_2$	-Γ$_1$	-I	U$_3$	- U$_2$	U$_1$	-Θ$_1$	V$_3$	- V$_2$	V$_1$	-Θ$_2$	-W$_3$	W$_2$	-W$_1$	Θ$_3$
Θ$_1$	-U$_1$	- U$_2$	-U$_3$	-I	Γ$_1$	Γ$_2$	Γ$_3$	Θ$_3$	W$_1$	W$_2$	W$_3$	–Θ$_2$	--V$_1$	-V$_2$	-V$_3$

U_1	Θ_1	$-U_3$	U_2	$-\Gamma_1$	-I	$-\Gamma_3$	Γ_2	W_1	$-\Theta_3$	W_3	$-W_2$	V_1	$-\Theta_2$	V_3	$-V_2$
U_2	U_3	Θ_1	$-U_1$	$-\Gamma_2$	Γ_3	-I	$-\Gamma_1$	W_2	$-W_3$	$-\Theta_3$	W_1	V_2	$-V_3$	$-\Theta_2$	V_1
U_3	$-U_2$	U_1	Θ_1	$-\Gamma_3$	$-\Gamma_2$	Γ_1	-I	W_3	W_2	$-W_1$	$-\Theta_3$	V_3	V_2	$-V_1$	$-\Theta_2$
Θ_2	$-V_1$	$-V_2$	$-V_3$	$-\Theta_3$	$-W_1$	$-W_2$	$-W_3$	-I	Γ_1	Γ_2	Γ_3	Θ_1	U_1	U_2	U_3
V_1	Θ_2	$-V_3$	V_2	$-W_1$	Θ_3	W_3	$-W_2$	$-\Gamma_1$	-I	$-\Gamma_3$	Γ_2	$-U_1$	Θ_1	U_3	$-U_2$
V_2	V_3	Θ_2	$-V_1$	$-W_2$	$-W_3$	Θ_3	W_1	$-\Gamma_2$	Γ_3	-I	- $-\Gamma_1$	$-U_2$	$-U_3$	Θ_1	U_1
V_3	$-V_2$	V_1	Θ_2	$-W_3$	W_2	$-W_1$	Θ_3	$-\Gamma_3$	$-\Gamma_2$	Γ_1	-I	$-U_3$	U_2	$-U_1$	Θ_1
Θ_3	W_1	W_2	W_3	Θ_2	$-V_1$	$-V_2$	$-V_3$	$-\Theta_1$	U_1	U_2	U_3	-I	$-\Gamma_1$	$-\Gamma_2$	$-\Gamma_3$
W_1	$-\Theta_3$	W_3	$-W_2$	V_1	Θ_2	V_3	$-V_2$	$-U_1$	$-\Theta_1$	U_3	$-U_2$	Γ_1	-I	Γ_3	$-\Gamma_2$
W_2	$-W_3$	$-\Theta_3$	W_1	V_2	$-V_3$	Θ_2	V_1	$-U_2$	$-U_3$	$-\Theta_1$	U_1	Γ_2	$-\Gamma_3$	-I	Γ_1
W_3	W_2	$-W_1$	$-\Theta_3$	V_3	V_2	$-V_1$	Θ_2	$-U_3$	U_2	$-U_1$	$-\Theta_1$	Γ_3	Γ_2	$-\Gamma_1$	-I

Table 5: Multiplication Table Of Sixteen Sedenion Basis Elements Of I, Γ_1, Γ_2, Γ_3, Θ, U_1, U_2, U_3, Θ, V_1, V_2, V_3, Θ, W_1, W_2, And W_3

The above table display one quaternion for the 4D exterior spacetime and three 4-element sets, U_k, V_k and W_k, for the 12D internal spacetime with each quartet containing own spinor triplet and a pseudo time operators The sedenion algebra has a richer structure, it contains three sets of suboctonion algebra with a commonly shared the same quaternion algebra of $\{\mathbf{I}, \mathbf{\Gamma}_1, \mathbf{\Gamma}_2, \mathbf{\Gamma}_3\}$, which represents the exterior spacetime, The inner spacetime consists of three quaternion-like quartets $\{\mathbf{\Theta}_1, \mathbf{U}_1, \mathbf{U}_2, \mathbf{U}_3\}$, $\{\mathbf{\Theta}_2, \mathbf{V}_1, \mathbf{V}_2, \mathbf{V}_3\}$ and $\{\mathbf{\Theta}_3, \mathbf{W}_1, \mathbf{W}_2, \mathbf{W}_3\}$ with each quartet contains a cyclic spinor triplet. These three quartets in the sedenion algebra e due to the split of the quartet in the octonion algebra represent the internal spacetime and play a key role in three generations of leptons and quarks. To derive the fine-structure constant, the 12 basis elements of the sedenion algebra will be used to represent the generalized gauge operators, equivalently, the rest mass operator. There are three internal quartets for the internal spacetime as illustrated in Fig. 1, there are four cyclic sponsor triplets represented by four sets of SU (2) generators as shown in Fig. 2. Therefore, there are 12 degrees of freedom in the electron's internal spacetime, i.e., three quaternion-like quartets and three cyclic spinor triplets, and a total of 12 anti-commutative operators to represent the internal spacetime.

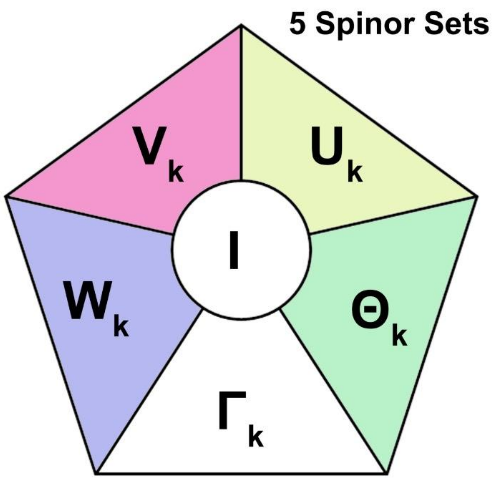

Figure 2: The Schematic Diagram Shows that the Sedenion Algebra Contains an Identity Element and Five Cyclic Spinor Triplet Sets

I and $\mathbf{\Gamma}_k$ represents the exterior spacetime, while $\mathbf{\Gamma}_k$ represents pseudo time spinor triplet, U_k, V_k, and W_k represent the spatial spinor triplets in the internal spacetime.
The internal spacetime consists of four sets of the SU (2) spinor triplets.

Following the development strategy for the octonion algebra, we consider

$$\mathbf{M} = \mathbf{\Theta} + \mathbf{P}$$

$$\mathbf{\Theta} \equiv \sum_{k=1}^{3} \theta_k \mathbf{\Theta}_k \, , \mathbf{P} \equiv \sum_{k=1}^{3} \mathbf{P}_k$$

$$\boldsymbol{P}_1 \equiv \sum_{k=1}^{3} P_{1,k}\mathbf{U}_k, \mathbf{P}_2 \equiv \sum_{k=1}^{3} P_{2,k}\mathbf{V}_k, \mathbf{P}_3 \equiv \sum_{k=1}^{3} P_{3,k}\mathbf{W}_k \quad (5A)$$

$$-\boldsymbol{m}_0{}^2|\Psi\rangle = \mathbf{M}^2|\Psi\rangle = (\mathbf{\Theta}^2 + \mathbf{P}^2)|\Psi\rangle = -\left(\sum_{k=1}^{3}\theta_k{}^2 + \sum_{j,k=1}^{3}P_{j,k}{}^2\right)|\Psi\rangle,$$

$$\mathbf{P}_1{}^2|\Psi\rangle = -\sum_{k=1}^{3}P_{1,k}{}^2|\Psi\rangle, \mathbf{P}_2{}^2|\Psi\rangle = -\sum_{k=1}^{3}P_{2,k}{}^2|\Psi\rangle, \mathbf{P}_3{}^2|\Psi\rangle = -\sum_{k=1}^{3}P_{3,k}{}^2|\Psi\rangle$$

As shown in Table 5, there are five cyclic spinor sets $\{\mathbf{\Theta}_1, \mathbf{\Theta}_2, \mathbf{\Theta}_3\}$,$\{\mathbf{U}_1, \mathbf{U}_2, \mathbf{U}_3\}$, $\{\mathbf{V}_1, \mathbf{V}_2, \mathbf{V}_3\}$, and $\{\mathbf{W}_1, \mathbf{W}_2, \mathbf{W}_3\}$, which anti-commute with each other. From the above equation, one obtains

$$exp(\mathbf{m}\tau) = cos(\omega\tau) + (\mathbf{m}/\omega)\, sin(\omega\tau)$$

$$\omega^2 \equiv \sum_{k=1}^{3}\theta_k{}^2 + \sum_{k=1}^{3}P_{1,k}{}^2 + \sum_{k=1}^{3}P_{2,k}{}^2 + \sum_{k=1}^{3}P_{3,k}{}^2 \quad (5B)$$

The square-sum relation above represents the Pythagorean rule for the square of the rest mass energy of the electron.

According to the sedenion model, twelve basis elements represent the internal degrees of freedom for the gauge field to describe the rest mass. By mass quantization, the masses are not continuous but discrete, and the fundamental model frequency corresponds to a specific prime number. By imposing gauge quantization, or equivalently mass quantization, like the octonion model, we extend the criteria to twelve integers. After an extensive search, we found solutions where the prime decomposition of the integers satisfies the constraints imposed by the sedenion model. The resulting fine structure constant, derived from these quantized values, matches the experimental value remarkably well with 10-12 deviation. According to Eq. (5A) and following similar logical reasoning about gauge or mass quantization protocols: 1) $\mathbf{m} = \mathbf{\Theta} + \mathbf{P}$ is represented by an integer equation with a prime for both $\mathbf{m}$ and P; 2) $\mathbf{\Theta} = \sum k\theta k\mathbf{\Theta}k$represented by an integer triplet, 3)$\mathbf{P}k$= $P1, k\mathbf{U1}$+ $P2, k\mathbf{V}2$ + $P3, k\mathbf{W}K$, which are represented by integer triplets.
In Table 6, we summarized the criteria for allowable 12 integers.

$\boldsymbol{M} = \boldsymbol{P} + \mathbf{\Theta}$ $\zeta = P^{2^2} + \theta^2$ $\{\zeta: P, \theta\}$ $\zeta, P: primes$	$\mathbf{P} = \mathrm{P}_1 + \mathrm{P}_2 + \mathrm{P}_3$ $\mathbf{P}_1 = \sum_{k=1}^{3} P_{1,k}\mathbf{U}_k$ $\mathbf{P}_2 = \sum_{k=1}^{3} P_{2,k}\mathbf{V}_k$ $\mathbf{P}_3 = \sum_{k=1}^{3} P_{3,k}\mathbf{W}_k$	$\mathbf{\Theta} = \sum_{k=1}^{3}\theta_k\mathbf{\Theta}_k$ $\theta^2 =$ $\theta_1{}^2 + \theta_2{}^2 + \theta_3{}^2$ $\{\theta: \theta_1, \theta_2, \theta_3\}$	$\zeta, P: primes$ $other integers > 1$ $\lvert mod(R/\zeta, \pi)\rvert < 10^{-2}$ $R = \frac{1}{\zeta}\left(\prod_{j,k=1}^{3}P_{j,k}\right)\left(\prod_{k=1}^{3}\theta_k\right)$ $\zeta = \sum_{j,k=1}^{3}P_{j,k}{}^2 + \sum_{k=1}^{3}\theta_k{}^2$

Table 6: Quantized Hyperfine Gauge For The Effective Rest Mass And Constraints On 12 Integers, Representing Four Triplets Of SU (2) Generators, Representing $(\mathrm{su}(2) \oplus \mathrm{su}(2) \oplus \mathrm{su}(2)) \times S_3 \oplus$ **Su (2) Algebra**

After an extensive computer searching for 12 integers that satisfy the above constraints on prime, triplet or doublet, we have identified a prime and decompositions into a sum over the squares of 12 non-zero integers as shown in Table 7. After search efforts using a computer, although we have found several Pythagorean primes that satisfy some of the criteria, such as 137035999201, 137035999209569, etc., most do not meet all requirements. Listed in Table 6 is the allowed set of integer square decomposition from the Pythagorean prime.

$P_1{}^2 = \sum_{k=1}^{3}P_{1,k}^2, P_2{}^2 = \sum_{k=1}^{3}P_{2,k}^2, P_3{}^2 = \sum_{k=1}^{3}P_{3,k}^2$	
$2nd - layer Triplet$ $\{P_{11,}P_{12,}P_{13}\},\{P_{21,}P_{22}P_{23}\},\{P_{31,}P_{32,}P_{33}\}$ $P_{1,k} = \{1,4,8\}$ $P_{2,k}, = \{8, 123966, 706992\}$ $P_{3,k} = \{14, 214096, 2466632\}$	$\theta_{k=}\{13360, 847299, 11378477\}$

$2nd-layerTriplet$ $\{P_{11,}P_{12,}P_{13}\},\{P_{21,}P_{22}P_{23}\},\{P_{31,}P_{32,}P_{33}\}$ $P_{1,k}=\{1,4,8\}$ $P_{2,k},=\{8,123966,706992\}$ $P_{3,k}=\{14,214096,2466632\}$	$\theta_{k=}\{13360,847299,11378477\}$

Table 7: List of the Pythagorean Prime and 12 Components, Representing 12 Elements of $(\mathrm{su}(2)\oplus\mathrm{su}(2)\oplus\mathrm{su}(2))\times S_3\oplus\mathrm{su}(2)$ **Algebra, According To the Sedenion Model**

In the above table, the decomposition of the Pythagorean prime yields a prime P and an even integer Q. P square can be decomposed into a triplet, and each can be further decomposed into its own triplet. Therefore, P can be decomposed into three tiers of triplets, plus the decomposition of Q square into a triplet, one obtains $\mathbf{su}\ (\mathbf{2})\oplus\mathbf{su}\ (\mathbf{2})\oplus\mathbf{su}\ (\mathbf{2})\oplus\mathbf{su}\ (\mathbf{2})$ algebra, which is applicable to leptons and not quarks.

Because in the equations involving only integers for quantized internal spacetime, we reduce mass, energy, and momentum, in units of $\Omega_0$and Ω_S, the fundamental frequency for the octonion and sedenion model, respectively, we need to relate these dimensionless integers to physical quantities. According to Table 6 and Eq. (5B), one has $\omega^2/\Omega S^2$ = = 137035999206077, By comparing it to the result from the octonion model with $m0^2/\Omega 0^2$ = 137, the fundamental mass $m_0{}^2/\Omega_S{}^2=\zeta,\Omega_S/\Omega_0=\sqrt{137/\zeta}\approx 1.00013\times 10^{-6}$. Because these primes numbers of 137 for the octonion model and for the sedenion model are dimensionless numbers, to relate them to physical quantity one makes a link of 137 to the electron mass. Thus, Ω_S a fundamental mass energy of about 0.5 eV, could be related to the mass of neutrinos. One can

scale down the integers in Table 3 by a factor of $\Omega_S/\Omega$for all decomposed components, we obtain, according to our sedenion approach, the theoretical value of $1/\alpha$ = 137.035999206077,which matches the experimental value of 137.035999206 with an unprecedented small discrepancy of ~ 10-12.

Conclusions

In this work, we explore the rich mathematical structures of hypercomplex numbers and their applications in physics. Using the Cayley-Dickson construction, we extend real numbers to complex numbers, then systematically build higher-dimensional hypercomplex systems— quaternions, octonions, and sedenions. We first examine quaternion applications in electromagnetism, special relativity, and massless relativistic fermions. Next, we show that the Dirac equation, traditionally described with four anti-commutative gamma matrices, can also be formulated using five of the seven anti-commutative imaginary operators in octonion algebra. We then propose a generalized Dirac equation by extending the conventional Lorentz scalar gauge to quaternionic, octonionic, and sedenionic gauges. This hypercomplex gauge naturally breaks Lorentz symmetry, leading to the mass generation for electrons and other particles. Through gauge quantization, we extend the scalar gauge to a quaternion gauge, resulting in integer equations (Table 4) that correspond to the quantization of effective mass via internal structural dynamics represented by octonionic anti-commutative operators. As shown in Sec. 2.3, we derive the inverse fine-structure constant as 137—a Pythagorean prime satisfying all constraints in Table 4. An extensive computational search for primes below 105 confirms that 137 is the only viable solution.

We extend the octonion model to the sedenion model to improve the predicted value with the experiments. We show that on Sec. 2.4, according to the sedenion algebra, it contains four sets of SU (2) spinor triplets and a total of 12 degrees of freedom for the internal dynamics. This sedenion model as imposed by Eq. (6A). We derived a set of constraints on the Pythagorean prime and its decomposition to a set of integer squares, as shown in Table 6. After extensive computer screening, of possible solutions, we have found a Pythagorean prime, and its decomposed integer squares as listed in Table 7. According to our analysis, we obtain the theoretical value of $1/\alpha$ = 137.035999206077,which matches the experimental value of 137.035999206 with remarkable accuracy with only ~ 10-12 discrepancies. In our other work, we have found a tiny variation in the predicted value with $1/\alpha$ = 137.03599920605017 for u (1) ⊕ su(2) ⊕ su(3), instead of (su(2) ⊕ su(2) ⊕ su(2)) × S_3 ⊕ su(3) considered here. There is a small difference occurring at the 11th decimal digit. The difference might be related to the difference in the symmetry group, but such a small difference has no significant influence.

In summary, using hypercomplex algebra, we extend Dirac equations for the electron with quantized gauge fields. This approach allows us to derive the fine-structure constant with unprecedented accuracy theoretically. As a fundamental parameter in electromagnetism, it governs photonics, chemical bonding, molecular structures, and material properties. Understanding its origin deepens our insight into the interactions between photons and charged particles. In a related study, we also uncover intriguing links between the fine-structure constant and the mass ratios of electrons, leptons, and quarks, suggesting possible connections between electromagnetism and the other fundamental forces, warranting further investigation. Until now, there is no known theoretical work that could explain the origin of the fine structure constant, predict its value and explain why this constant is dimensionless Therefore, we think our approach in combining hypercomplex algebra and spacetime quantization can accomplish such goals.

We have recently extended the sedenion algebra and hypercomplex gauge theory k to explain the rise of three generations of leptons and quarks, the masses of these fermions, and the mass oscillations among three flavor neutrinos. Our proposed route offers an alternative beyond the Standard Model description, and could explain some unsolved problems in particle physics, unlike twistor theory, string theory, and loop quantum gravity. We believe such an approach could potentially lead to quantum gravity and the grand unification theory (GUT) of all four forces in nature. More details on our further development will be published elsewhere.

Author Contributions

J. T. initiated the project, conceived the model, derived the equations, and wrote the manuscripts.
Q. T. prepared the figures and reference list.

Data Availability Statement

This work contains mostly analytical equation derivations, and some computer codes for tests of whether they are prime were written in Fortran. Which are available upon reasonable request.

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