Archives of Interdisciplinary Education ISSN: 3068-9317

PRIME OPEN ACCESS

Volume 2, Issue 1

Research Article

Date of Submission: Dec 30, 2025
Date of Acceptance: Jan 21, 2026
Date of Publication: Feb 16, 2026

Solving Fermat Last Theorem and the Generalized Diophantine Equations

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Citation: Mantzakouras, N., Zapata, C. L. (2026). Solving Fermat Last Theorem and the Generalized Diophantine Equations. *Arch Interdiscip Educ*, *2*(1), 01-16.

Abstract
The Pythagorean theorem is perhaps the best known theorem in the vast world of mathematics; a simple relation of square numbers that encapsulates the glory of mathematical science. It is also justifiably the most popular but also the most sublime theorem in mathematical science. The starting point was *Diophantus' 20th problem* (*Book VI of the Arithmetica of Diophantus*), the generalized method we use here is based on elementary inequalities and gives a solution to any Diophantine equation of degree n with respect to the number of variables d. It is a method that does not exclude other equivalent analytic methods for finding specific solutions. However, the proposed method generalizes to any exponential positive integer $n \in \mathbb{Z}^+_{\geq 3}$, with number of variables $d \geq 3$. An explicit proof is also given for variables in both parts with equal or different numbers of variables, also of degree n. By this logic we can obtain many extensions to other diophantine equations that present great difficulty in being treated by diophantine analysis and number theory. As a direct application it is Fermat's Last Theorem and of course hasmuch broader generalization to symmetric or asymmetric Diophantine equations.

Introduction and Preliminaries

Theorem 1 [(14),(3), (7), (13)]
We consider the sequence of variables $x_1, x_2, x_3, \ldots, x_d$ such that they are integers that are different in general from each other and also that the equality

$$x_1^n + x_2^n + x_3^n + \cdots + x_{d-1}^n = x_d^n,$$

where $n \in \mathbb{N}_{>2}, and\, d \in \mathbb{N}_{>2}$ indicating the number of variables.
Prove the two basic inequalities:

(a) $\left(\frac{x_d}{x_{d-1}}\right)^n < d-1 < \left(\frac{x_d}{x_1}\right)^n$

(b) If $x_d = x_{d-1} + k, k \in \mathbb{N}+$ then $k < x_{d-1} - (d-1)$

Proof. (a) We assume that the equation is valid,(1.2)

$$x_1^n + x_2^n + x_3^n + \cdots + x_{d-1}^n = x_d^n \tag{T.1}$$

where $n, d \in \mathbb{N}_{>2}$. Assuming the general inequality order

$$x_1 < x_2 < x_3 < \cdots < x_{d-1} < x_d \Leftrightarrow x_1^n < x_2^n < x_3^n < \cdots < x_{d-1}^n < x_d^n, \quad n \in \mathbb{N}_{>2} \tag{T.2}$$

Combining relations (T.2) and (T.1), we obtain:

$$x_1^n + x_1^n + x_1^n + \cdots + x_1^n < x_1^n + x_2^n + x_3^n + \cdots + x_{d-1}^n = x_d^n$$

therefore we have

$$(d-1)x_1^n < x_d^n \Leftrightarrow (d-1) < \left(\frac{x_d}{x_1}\right)^n \tag{T.3}$$

With the same logic it will hold that:

$$x_{d-1}^n + x_{d-1}^n + x_{d-1}^n + \cdots + x_{d-1}^n > x_1^n + x_2^n + x_3^n + \cdots + x_{d-1}^n = x_d^n$$

therefore we have

$$(d-1)x_{d-1}^n > x_d^n \Leftrightarrow (d-1) > \left(\frac{x_d}{x_{d-1}}\right)^n \tag{T.4}$$

If we put relations (T.3 and T.4) together we get the obvious but very elementary and at the same time important relation:

$$\left(\frac{x_d}{x_{d-1}}\right)^n < d-1 < \left(\frac{x_d}{x_1}\right)^n \tag{T.5}$$

(b) To complete the proof, we proceed with very simple yet crucial inequality relations that will be used for the final Theorem 2. We consider the validity of the inequality

$$x_d = x_{d-1} + k, k \in \mathbb{N}^+ \text{ then } k < x_{d-1} - (d-1), \text{ where } d \in \mathbb{N}_{>2}. \tag{T.6}$$

The proof follows two different cases,

1. We consider the extreme case if $x_1 > k \geq 1$ where $d \in \mathbb{N}_{>2}$

We assume $x_1 > k \geq 1$, where $d \in \mathbb{N}_{>2}$. Consequently, the sequence satisfies:

$$x_d > x_{d-1} > \cdots > x_2 > x_1 > k \geq 1 \tag{T.7}$$

From relation (T.7), the following inequalities directly result:

$$x_{d-1} \geq x_{d-2} + 1, \quad x_{d-2} \geq x_{d-3} + 1, \quad \ldots, \quad x_2 \geq x_1 + 1, \quad x_1 \geq k + 1 \tag{T.8}$$

Summing these inequalities gives us the relevant general relation: $x_{d-1} \geq k+(d-1) \Leftrightarrow k \leq x_{d-1} - (d-1)$ (T.9), applicable when $d \geq 3$.

We aim to demonstrate that $k \neq x_{d-1} - (d-1)$ (T.10), given our assumption that $d \in \mathbb{N}_{>2}$. For equality to hold universally, we must have $k = x_1 - 1 = x_2 - 2 = \ldots = x_{d-1} - (d-1)$ (T.11). In a more general form, this follows from (T.11) that $k = x_i - i = x_{d-1} - (d-1)$, where $1 \leq i \leq d-1$ (T.12).

However, (T.12) leads to the basic relation that identifies an incorrect logic: $1 \leq x_i - x_{d-1} + d - 1 \leq 0 + d - 1$ or $1 \leq d - 1 \Leftrightarrow d \geq 2$. This appears to contradict the hypothesis, since we have assumed $d \in \mathbb{N}_{>2}$ or $d \geq 3$.

2. We consider the intermediate case $x_i \leq k \leq x_{d-1}$ or $x_{i-1} \leq k < x_i$ or $2 \leq i \leq d-1$

According to the preceding discussion and relation (T.8), the following inequality relations hold:

$$x_{d-1} \geq x_{d-2} + 1, \quad x_{d-2} \geq x_{d-3} + 1, \quad \ldots, \quad x_2 \geq x_1 + 1, \quad x_1 \geq k + 1 \tag{1.3}$$

Furthermore, as previously established, these sum up to the inequality $x_{d-1} \geq k + (d-1) \Leftrightarrow k \leq x_{d-1} - (d-1)$.

If we assume the inequality:

$$x_{d-1}-(d-1) \leq k \leq x_{d-1}+(d-1) \Leftrightarrow k \leq x_{d-1}-(d-i) \text{ for } -(d-1) \leq (d-i) \Leftrightarrow 1 \leq i \tag{T.13}$$

However, the crucial inequality $1 \leq i$ contradicts our initial assumption, as $2 \leq i \leq d-1$. Hence, the relation $k \neq x_{d-1} - (d-1)$ (T.14) holds, and only the relation of interest remains valid: $k < x_{d-1} - (d-1)$ (T.15).

Thus, we have established two fundamental inequalities that lead us to the final set of inequalities and indicate the minimum value of the exponent n in the generalized Diophantine equation:

$$x_1^n + x_2^n + x_3^n + \cdots + x_{d-1}^n = x_d^n, \quad \text{where } d \in \mathbb{N}_{>2}. \qquad (1.4)$$

This equation seeks to determine the restrictive integer limit of the exponent n, so that there are solutions to this Diophantine equation. Through this straightforward yet concise logic and procedure, we can solve this very difficult problem, for which no clear answer has yet been given, except for the case d = 3 and n ≥ 3, by various methods, generally called the proof of Fermat's Last Theorem Fermat.

Main Results

Theorem 2 . The count of solutions for the exponent n in the equation

$$x_1^n + x_2^n + x_3^n + \cdots + x_{d-1}^n = x_d^n,$$

where $d \in \mathbb{N}_{>2}$, is determined by the equality $n_{down} = \left\lfloor \frac{\log(d-1)}{\log\left(\frac{d-1}{d}\right)} \right\rfloor$ as a function of d (where ⌊·⌋ denotes the floor function or integer part, and log denotes the natural logarithm).

Therefore, for values of n exceeding the upper bound, i.e., for

$$n_{up} = \left\lfloor \frac{\log(d-1)}{\log\left(\frac{d-1}{d}\right)} \right\rfloor + 1, \qquad (2.1)$$

the Diophantine equation has no solutions.
Proof.

A) According to Theorem 1, two basic relations will hold.

$$(a) \qquad \left(\frac{x_d}{x_{d-1}}\right)^n < d-1 < \left(\frac{x_d}{x_1}\right)^n \qquad (2.2)$$

$$(b) \qquad \text{If } x_d = x_{d-1} + k, k \in \mathbb{N}^+ \text{ then } k < x_{d-1} - (d-1) \qquad (2.3)$$

For the equation $x_1^n + x_2^n + x_3^n + \cdots + x_{d-1}^n = x_d^n$, where $d, \ n \in \mathbb{N}_{>2}$. If we isolate the first part of the first relation i.e.,

$$\left(\frac{x_d}{x_{d-1}}\right)^n < d-1 \quad (\Theta.1)$$

and the second part $k < x_{d-1} - (d-1)$ with $x_d = x_{d-1} + k, \ k \in \mathbb{N}^+ (\Theta.2)$, then we will obtain two inequalities:

$$(d-1)^{\frac{1}{n}} - 1 < \frac{k}{k+d-1} \quad (\Theta.3)$$

$$(d-1)^{\frac{1}{n}} - 1 > \frac{k}{k+d-1}. \quad (\Theta.4)$$

To determine which of the two inequalities is acceptable for n, we need to make a choice. The previous relations for n ultimately take the following forms, involving the integer part:

$$n \geq \left\lfloor \frac{\log(d-1)}{\log\left(\frac{2k+d-1}{d+k-1}\right)} \right\rfloor + 1 \quad (\Theta.5)$$

$$n \leq \left\lfloor \frac{\log(d-1)}{\log\left(\frac{2k+d-1}{d+k-1}\right)} \right\rfloor \quad (\Theta.6)$$

But of course, we want the cases where we have $k = 1$ because this will result in the minimum value for the content of the integer part. This can be seen clearly if we take

$$\varepsilon = \frac{\log(d-1)}{\log\left(\frac{2k+d-1)}{d+k-1}\right)} = \frac{\log(d-1)}{\log(1+\frac{k}{d+k-1})} = \frac{\log(d-1)}{\log\left(1+\frac{1}{1+\frac{d-1}{k}}\right)} \quad (\Theta.7)$$

and call the value for ε becomes minimum when $k = 1$ and will then take the value

$$\varepsilon_{\min} = \frac{\log(d-1)}{\log\left(1+\frac{1}{d}\right)} = \frac{\log(d-1)}{\log\left(\frac{d+1}{d}\right)} \quad (\Theta.8)$$

This will be the final minimum real value (Θ.8) we are interested in for the value of the exponent n, because it reduces to the minimum acceptable values it can take. Eventually, we will clearly get its integer value for the allowable value of the exponent. Any other value removes the exponent from the real values and therefore will not be the one we are looking for. We therefore obtain the final forms of the relations for the exponent n:

$$n_{\text{up}} = \lfloor \frac{\log(d-1)}{\log\left(\frac{d-1}{d}\right)} \rfloor + 1, \quad (\Theta.9)$$

$$n_{\text{down}} = \lfloor \frac{\log(d-1)}{\log\left(\frac{d-1}{d}\right)} \rfloor \quad (\Theta.10)$$

Sincewe have assumed k ≥ 1 and d ≥ 3 for the equation $x_1^n + x_2^n + x_3^n + \cdots + x_{d-1}^n = x_d^n$, where $d \in \mathbb{N}_{>2}$, in order for one of the two relations (Θ.9, Θ.10) to hold, it must include $n = 1$ or $n = 2$ which are known to be acceptable and valid. By this obvious logic, it obviously follows that the second inequality will hold and the first will not. The first will be the cases of n which are excluded to yield solutions for the Diophantine equation.

B) From relation (Θ · 8), which refers to the integer minimum content.

$$\varepsilon_{\min} = \frac{\log(d-1)}{\log\left(\frac{d+1}{d}\right)} \quad (\Theta \cdot 11)$$

Of the above relations, we can prove that it is always a real number, namely $\varepsilon_{\min} \in \mathbb{R}^+ - \mathbb{Q}^+$. This means that in no case do the two limits of n coincide for every integer d. It is a correct expected result that rules out the possibility of an error in choosing the correct integer that will lead us to the existence of solutions and therefore this proof is absolutely useful. We will need a more understandable form to convert the natural logarithm to an integer base and let us choose 2.

We therefore start from relation Θ.11 and transform it:

$$\varepsilon_{\min} = \frac{\log(d-1)/\log 2}{\log\left(\frac{d+1}{d}\right)/\log 2} = \frac{\log_2(d-1)}{\log_2\left(\frac{d+1}{d}\right)} \qquad (\Theta.12)$$

Assuming the replacements

$$d-1 = 2^m \Leftrightarrow d = 1+2^m \text{ and therefore } \log_2(d-1) = m, m \in \mathbb{Z}^+.$$

Also, $\log_2\left(\frac{d+1}{d}\right) = \log_2\left(1+\frac{1}{d}\right) = \log_2\left(1+\frac{1}{2^m+1}\right)$. It will follow that

$$\varepsilon_{\min} = \frac{m}{\log_2\left(1+\frac{1}{2^m+1}\right)}, m \in \mathbb{Z}^+ \quad (\Theta.13)$$

We have to prove that $\log_2\left(1+\frac{1}{2^m+1}\right) \in \mathbb{R}^+ - \mathbb{Q}^+$ and then obviously it will hold that $\varepsilon_{\min} \in \mathbb{R}^+ - \mathbb{Q}^+$.

We will need two partial proofs. First, we will develop the relation $\log_2\left(1+\frac{1}{2^m+1}\right)$ in Maclaurin series. The expansion will be:

$$(2\log[2]-\log[3])-\frac{1}{6}\cdot\log[2]\cdot(m-1)+\frac{1}{72}\cdot(\log[2])^2\cdot(m-1)^2+$$
$$\frac{1}{81}\cdot(\log[2])^3\cdot(m-1)^3-\frac{11}{5184}\cdot(\log[2])^4\cdot(m-1)^4-\frac{1}{972}\cdot(\log[2])^5\cdot(m-1)^5+O[m-1]^6.$$

But the final relationship we are interested in $\log_2\left(1+\frac{1}{2^m+1}\right)$ will be in final form:

$$\frac{2\log[2]-\log[3]}{\log[2]}-\frac{m-1}{6}+\frac{1}{72}\cdot\log[2]\cdot(m-1)^2+$$
$$\frac{1}{81}\cdot(\log[2])^2\cdot(m-1)^3-\frac{11}{5184}\cdot(\log[2])^3\cdot(m-1)^4-\frac{1}{972}\cdot(\log[2])^4\cdot(m-1)^5+O[m-1]^6.$$

We will therefore have a sum of terms that has two constant integers and the other terms will be in log 3/ log 2 form and the other terms in log 2 form. But log 2 (see reference (2)) is an irrational number as is the ratio log 3/ log 2 (see (15)). So the whole sum of the series will be an irrational number and by extension the value of

$$\varepsilon_{\text{min}}=\frac{m}{\log_2\left(1+\frac{1}{2^m+1}\right)}, m\in\mathbb{Z}^+, \varepsilon_{\text{min}}\in\mathbb{R}^+-\mathbb{Q}^+,$$

as a ratio of an integer to an irrational value of a number.

Therefore, it will hold that $n_{\text{down}}\neq n_{\text{up}}, \forall d\in\mathbb{Z}^+$ with acceptable values of the exponent *n*, so that there is a solution to the generalized Diophantine equation

$$x_1^n+x_2^n+x_3^n+\cdots+x_{d-1}^n=x_d^n,$$

where $n\in\mathbb{N}_{>2}$ only for values of $n\leq n_{\text{down}}, \forall d\in\mathbb{Z}^+_{\geq 3}$. Obviously, for any arrangement with *d* number of variables, we will follow the same procedure with the corresponding logarithm base and come to the same conclusion, i.e., if $d\in\mathbb{Z}^+_{>2}, \varepsilon_{\text{min}}\in\mathbb{R}^+-\mathbb{Q}$.

(Theorem 3) FERMAT'S LAST THEOREM

For any integer *n* > 2, the equation $x_1^n+x_2^n=x_3^n$, where $n\in\mathbb{N}_{>2}$ has no positive integer solutions (4).

Proof. By Theorem 2 (Θ.11), the acceptable values for the exponent *n* will be given by the relation

$$n_{\text{down}}=\left\lfloor\frac{\log(d-1)}{\log\left(\frac{d-1}{d}\right)}\right\rfloor, \quad d=3.$$

where ⌊.⌋ the integer part in this analysis.

For this particular Diophantine equation we will have

$$\varepsilon_{\text{min}}=\frac{\log_2(d-1)}{\log_2\left(\frac{d+1}{d}\right)}=2.409421$$

and hence according to the minimum value the upper value for *n* will be fully defined

$$n_{\text{down}}=\left\lfloor\frac{\log(d-1)}{\log\left(\frac{d-1}{d}\right)}\right\rfloor=\lfloor 2.409421\rfloor=2$$

The forbidden values of *n* will therefore be

$$n\geq\left\lfloor\frac{\log(d-1)}{\log\left(\frac{d-1}{d}\right)}\right\rfloor+1=\lfloor 2.409421\rfloor+1=3, \quad n\geq 3$$

Therefore, Fermat's Diophantine equation does not have any solutions for $n \geq 3$. The proof with the given values is both complete and understandable, and it is also determinative for the values of n.

Indicative Values of the Allowed Values of *N*, In Relation to the Number of Variables *D* in the Eqation

A concise procedure with logical inequalities therefore provides a complete picture of the admissible solutions for n, without the need to analyze each case separately using complex and time-consuming methods, especially as the number of variables increases. The key to the calculation is obviously the value of

$$\varepsilon_{\min} = \frac{\log_2(d-1)}{\log_2\left(\frac{d+1}{d}\right)}, d \in \mathbb{Z}^{+}{}_{>2}$$

from (Θ.11) and the two values n_{up} and n_{down} as defined in Theorem 2.

The values of n_{down}, n_{up} which determine when the Diophantine equation has a solution and when it does not for $3 \leq d \leq 15$

d	$\varepsilon_{\min}$	n_{down}	n_{up}
3	2.40942084	2	3
4	4.923343212	4	5
5	7.603568034	7	8
6	10.44067995	10	11
7	13.41826	13	14
8	16.52114108	16	17
9	19.73644044	19	20
10	23.05340921	23	24
11	26.46303475	26	27
12	29.95769812	29	30
13	33.53089523	33	34
14	37.17701992	37	38
15	40.89119619	40	41

Table 1: The values of n_{up}, n_{down}

Therefore for $1 \leq n \leq n_{down}$,we always have solutions for the generalized Ferm Diophantine equation while for indices $n \geq n_{up}$ the Fermat equation cannot possibly have a solution and therefore any attempt to find solutions will be in vain for the variables x_i, where $i \in \mathbb{N}^{+}_{>2}$ $d > 2$ denotes the number of variables. For example, cases that have been considered include $d = 3$ (the well-known case of Fermat), while for d = 4, it has been shown not to exist for $n > 4$ (9) (Lander et al., 1967). For $d = 5$, instances such as $27^5 + 84^5 + 110^5 + 133^5 = 144^5$ can be observed.

Lander and Parkin (1967)(8), Lander et al. (1967) (9), and Ekl (1998) (5) have made important contributions to the theory of which cases are solvable. According to the aforementioned theory, the range can go up to $n = 7$. For $n = 6$, cases with $d = 7, 8, 9$, and 10 are known. They also propose that $1 \leq n \leq n_{down} = 7$ if $d = 5$. However, such cases have not yet been discovered. Thus, the theory remains valid for $d = 3$ and $d = 4$ practically and theoretically up to $n = 6$, as has been proved. Of course, the above proof makes perfect sense with the inequalities as defined works correctly.

Theorem 4

We consider the sequence of variables $x_1, x_2, x_3, \ldots, x_d$ such that they are integers, generally distinct from each other. Additionally, we consider the equality:

$$x_1^n + x_2^n + x_3^n + \ldots + x_{d-s-1}^n = x_{d-s}^n + x_{d-s+1}^n + \ldots + x_d^n$$

where $\{n \in \mathbb{N}_{>2}, d \in \mathbb{N}_{>3}\}$, and s is a finite integer indicating the number of variables in the right part excluding x_d.

Then, prove the following two basic inequalities:

(1) $\left(\frac{x_d}{x_{d-1}}\right)^n < d - 1 - 2s < \left(\frac{x_d}{x_1}\right)^n$,

- if $d = 2m$, then $1 \leq s \leq \frac{d-2}{2}$,

- or if $d = 2m + 1$, then $1 \leq s \leq \frac{d-3}{2}$,

where $m \in \mathbb{N}^+$ and $m \geq 2$.

(2) If $x_d = x_{d-1} + k$, $k \in \mathbb{N}^+$, then $k < x_{d-1} - (d-1)$.

Proof.

(1) We assume the validity of the equation: $x_1^n + x_2^n + x_3^n + \ldots + x_{d-s-1}^n = x_{d-s}^n + x_{d-s+1}^n + \ldots + x_d^n$ (T'.1) where $n, d \in \mathbb{N}_{>2}$. Assuming in general that the inequality $x_1 < x_2 < x_3 < \ldots < x_{d-1} < x_d$ holds if and only if $x_1^n < x_2^n < x_3^n < \ldots < x_{d-1}^n < x_d^n$, where $n \in \mathbb{N}_2^+$ (T'.2).

Combining relations (T'.2 & T'.1), we obtain:

$$x_1{}^n + x_1{}^n + x_1{}^n + \ldots + x_1{}^n - s \cdot x_1{}^n < x_1{}^n + x_2{}^n + x_3{}^n + \ldots + x_{d-s-1}{}^n - (x_{d-s}{}^n + \ldots + x_{d-1}{}^n) = x_d{}^n$$

therefore, we have $(d-1-2s)x_1{}^n < x_d{}^n$ if and only if $(d-1-2s) < \left(\frac{x_d}{x_1}\right)^n$ $(T'.3)$.

With the same logic, it will hold that

$$x_{d-1}^n + x_{d-1}^n + x_{d-1}^n + \ldots + x_{d-1}^n - s \cdot x_{d-1}^n > x_1^n + x_2^n + x_3^n + \ldots + x_{d-s-1}^n - (x_{d-s}^n + \ldots + x_{d-1}^n) = x_d^n$$

Therefore, we have $(d-1-2s)x_{d-1}^n > x_d^n \iff (d-1-2s) < \left(\frac{x_d}{x_{d-1}}\right)^n$ (T'.4).

If we put relations (T'.3 & T'.4) together, we get the obvious but very elementary and at the same time important relation:

$$\left(\frac{x_d}{x_{d-1}}\right)^n < d - 2s - 1 < \left(\frac{x_d}{x_1}\right)^n \quad \text{(T'.5)}$$

For inequalities (T'.5) to hold, the following conditions must be met:

(i) If $d = 2m$, then $1 \leq s \leq \frac{d-2}{2}$, because $d - 2s - 1 \geq 1$, $m \in \mathbb{N}_{\geq 2}^+$.

(ii) If $d = 2m+1$, then $1 \leq s \leq \frac{d-3}{2}$, because $d - 2s - 1 \geq 2$, $m \in \mathbb{N}_{\geq 2}^+$, which are easily demonstrated by substituting permissible cases.

b) To complete the proof, we continue with very simple inequality relations but very important ones, which we will use for the final theorem 2 that will follow. We consider the validity of the inequality $x_d = x_{d-1} + k$, $k \in \mathbb{N}^+$, then $k < x_{d-1} - (d-1)$, where $d \in \mathbb{N}_{\geq 3}$ (T'.6). This is true according to Theorem 1, case b.

Theorem 5

The number of existing solutions for the exponent n for the equation $x_1^n + x_2^n + x_3^n + \ldots + x_{d-s-1}^n = x_{d-s}^n + x_{d-s+1}^n + \ldots + x_d^n$, where $n \geq 3$, $d > 3$ positive integers is given by the relation

$$n_{\text{down}} = \left\lfloor \frac{\log(d-2s-1)}{\log\left(\frac{d+1}{d}\right)} \right\rfloor$$

as a function of d and s, where $\lfloor \cdot \rfloor$ denotes the integer part of a number and $\log$ is the natural logarithm.

Therefore, for values of n above the upper bound, i.e., for $n_{\text{up}} = \left\lfloor \frac{\log(d-2s-1)}{\log\left(\frac{d+1}{d}\right)} \right\rfloor + 1$, the Diophantine equation has no solution.

Proof. For the equation

$$x_1^n + x_2^n + x_3^n + \ldots + x_{d-s-1}^n = x_{d-s}^n + x_{d-s+1}^n + \ldots + x_d^n$$

where $n \geq 3$, $d > 3$ positive integers, if we isolate the first part of the first relation i.e.,

$$\left(\frac{x_d}{x_{d-1}}\right)^n < d-2s-1 \quad (\Theta'.1)$$

and the second part $k < x_{d-1}-(d-1)$ with $x_d = x_{d-1}+k$, $k \in \mathbb{N}^+$ $(\Theta'.2)$, then we will get two inequalities:

$$(d-2s-1)^{\frac{1}{n}} - 1 < \frac{k}{k+d-1} \quad (\Theta'.3)$$

$$(d-2s-1)^{\frac{1}{n}} - 1 > \frac{k}{k+d-1} \quad (\Theta'.4)$$

For this, we need to decide which of the two inequalities will be acceptable for n. The previous relations result in the following forms for n, associated with the integer part, which we take:

$$n \geq \left\lfloor \frac{\log(d-2s-1)}{\log\left(\frac{2k+d-1}{d+k-1}\right)} \right\rfloor + 1 \quad (\Theta'.5)$$

$$n \leq \left\lfloor \frac{\log(d-2s-1)}{\log\left(\frac{2k+d-1}{d+k-1}\right)} \right\rfloor \quad (\Theta'.6)$$

But of course, we want the cases where we have $k=1$ because this will result in the minimum value for the content of the integer part. This can be seen clearly if we take the content of the integer value and call

$$\varepsilon = \frac{\log(d-2s-1)}{\log\left(\frac{2k+d-1}{d+k-1}\right)} = \frac{\log(d-2s-1)}{\log\left(1+\frac{k}{d+k-1}\right)} = \frac{\log(d-2s-1)}{\log\left(1+\frac{1}{1+\frac{d-1}{k}}\right)} \quad (\Theta'.7)$$

The value for ε becomes minimum when $k=1$ and will then take the value:

$$\varepsilon_{min} = \frac{\log(d-2s-1)}{\log\left(1+\frac{1}{d}\right)} = \frac{\log(d-2s-1)}{\log\left(\frac{d+1}{d}\right)} \quad (\Theta'.8)$$

This will be the final minimum real value $(\Theta'.8)$ we are interested in for the value of the exponent n, because it reduces to the minimum the acceptable values it can take. We, therefore, obtain the final forms of the relations for the exponent n.

$$n_{up} = \left\lfloor \frac{\log(d-2s-1)}{\log\left(\frac{d+1}{d}\right)} \right\rfloor + 1 \quad (\Theta'.9)$$

$$n_{down} = \left\lfloor \frac{\log(d-2s-1)}{\log\left(\frac{d+1}{d}\right)} \right\rfloor \quad (\Theta'.10)$$

Lemma 1.

The symmetric equation

$$x_1^n + x_2^n + x_3^n + \ldots + x_{d-s-1}^n = x_{d-s}^n + x_{d-s+1}^n + \ldots + x_d^n \quad (\Theta'.11)$$

where $n \geq 0$, $d = 2m$, *and* $m \in \mathbb{N}^+_{\geq 2}$, *always has a solution. The number of existing solutions of the exponent* n *for the equation is infinite if* $n = 0$.

Proof.

(i) The value of s in this case is $s = \frac{d}{2} - 1$ therefore from relations $\Theta.9$ and $\Theta.10$ we obtain the same value for $n_{\text{down}} = n_{\text{up}} = 0$, because $\log(d-2s-1) = \log(d-(\frac{d}{2}-1)-1) = \log(1) = 0$ - Therefore apply only for $n = n_{\text{down}} = 0, n \in \mathbb{N}_0^+$ will always have a solution and infinity.

(ii) In each case, we add two variables to both sides and equate them with a variable raised to the n^{th} power. Thus, we arrive at a Diophantine equation of n^{th} degree with a total of d+2 variables. Otherwise, it is possible to consider the lowest category where we add one variable, i.e., we have d+1 variables.

i) Example: $2^0 + 3^0 + 6^0 = 7^0 + 8^0 + 10^0$

ii) Example: $x_1^n + x_2^n = x_3^n + x_4^n \Leftrightarrow x_1^n + x_2^n + x_e^n = x_3^n + x_4^n + x_e^n = x_v^n$. That is, once we add two variables integers x_e^n and x_v^n and we come to the upper minimum Diophantine n^{th} power. According to Table 1 (click here to refer to that table) in section IV page 7, then $n \leq 4$ corresponding to $d = 4$. Here we had to add 2 variables because of the weakness for $n = 3$. Otherwise, it is possible to have the lowest category, i.e., $d + 1$.

Theorem 6

The set of integer solutions (x, y, z) of the Diophantine equation $x^2 + y^2 = z^2$ (K.1) is given by the formula $(x, y, z) = (\pm t(a^2 - b^2), \pm 2abt, \pm t(a^2 + b^2))$ (K.2) or formula $(x, y, z) = (\pm 2abt, \pm t(a^2 - b^2), \pm t(a^2 + b^2))$ (K.3), where (α, b, t) are integer parameters such that $(\alpha, b) = 1$ and $a \neq b \pmod 2$ (17).

Proof. Assume that (x, y, z) is an integer solution of (K.1) different from $(0, 0, 0)$ and let us write $(x, y) \neq (0, 0)$ and $(x, y) = t$. Then $t \mid z$, and therefore, $(x, y, z) = (t\xi, t\eta, t\zeta)$ by $\xi^2 + \eta^2 = \zeta^2$ (K.4) where $(\xi, \eta) = 1$.

Therefore we have to solve the above relation and exclude the case $\xi \equiv \eta \equiv 0 \pmod 2$ will further imply that $(\zeta, \xi) = (\zeta, \eta) = 1$. In any solution of (K.4), the integers (ξ, η) cannot both be odd because then they would be $\xi^2 + \eta^2 \equiv 2 \pmod 4$, which is impossible since $\zeta^2 \equiv 1/0 \pmod 4$ for each integer ζ. Also, the fact that both of them are even is excluded from $(\xi, \zeta) = 1$.

So consider a solution where ξ is odd and η is even; therefore, ζ is odd. Then from (K.4), we get $\eta^2 = \zeta^2 - \xi^2 = (\zeta - \xi)(\zeta + \xi)$ with GCD of $(\zeta - \xi, \zeta + \xi)$ will divide $\zeta - \xi + \zeta + \xi = 2\zeta$ as well as $\zeta + \xi - (\zeta - \xi) = 2\xi$, from which it follows that $(\zeta - \xi, \zeta + \xi) = 1$ or 2.

Since $(\xi, \zeta) = 1$ but ξ and ζ are both odd, so $\zeta - \xi$ and $\xi + \zeta$ are both even, and therefore $(\zeta - \xi, \zeta + \xi) = 2$. Thus, we get the forms

$$(\frac{\eta}{2})^2 = \frac{\zeta^2 - \xi^2}{4} = \frac{(\zeta - \xi)}{2}\frac{(\zeta + \xi)}{2}$$

with $(\frac{\zeta+\xi}{2}, \frac{\zeta-\xi}{2}) = 1$.
Therefore, it will be

$$\frac{\zeta + \xi}{2} = \alpha^2 \quad \text{and} \quad \frac{\zeta - \xi}{2} = b^2 \quad \text{or} \quad \frac{\zeta + \xi}{2} = -\alpha^2 \quad \text{and} \quad \frac{\zeta - \xi}{2} = -b^2$$

where α, b are integers with $(\alpha, b) = 1$. From these imply that

$$\left(\frac{\eta}{2}\right)^2 = \frac{\zeta^2 - \xi^2}{4} = \frac{(\zeta - \xi)(\zeta + \xi)}{4} = \left(\frac{\zeta - \xi}{2}\right)\left(\frac{\zeta + \xi}{2}\right) = \alpha^2 b^2$$

or

$$\left(\frac{\eta}{2}\right)^2 = \alpha^2 b^2 \iff \eta^2 = 4\alpha^2 b^2 \iff \eta = \pm 2\alpha b \text{ or } \eta = \pm 2\alpha b.$$

It follows from the relation $(x, y) = t$ that there are two general forms for x, y, z:

$$(x, y, z) = \pm(t(a^2 - b^2), \pm 2abt, \pm t(a^2 + b^2))$$

(K.2)

and due to the symmetry of the equation K.1 will apply:

$$(x, y, z) = (\pm 2abt, \pm t(a^2 - b^2), \pm t(a^2 + b^2))$$

(K.3)

The triad (x, y, z) is called the Pythagorean triad because it is directly related to the Pythagorean theorem.

Proof of Fermat's Last Theorem Using Elementary Analysis

Theorem 7: Fermat's Last Theorem

If $n > 2$, the Diophantine equation $x^n + y^n = z^n$ (K.5) has no integer solutions (x, y, z) with $x \cdot y \cdot z \neq 0$.

Proof.

I. The case when $n = 2k, k \in \mathbb{Z}^+_{>1}$. From equation (K.5) for an integer exponent, it follows that since

$$x^{2k} + y^{2k} = z^{2k} \iff (x^k)^2 + (y^k)^2 = (z^k)^2$$

we are talking about positive values and we will have two symmetric cases:

$$(x^k, y^k, z^k) = \left(t(a^2 - b^2), 2abt, t(a^2 + b^2)\right) \quad \text{or}$$

$$(x^k, y^k, z^k) = \left(2abt, t(a^2 - b^2), t(a^2 + b^2)\right)$$

(K.6)

But the examination we are interested in will be limited to a single case due to symmetry. For the case of Fermat's theorem, when we want to prove what holds for positive even exponents, it is enough to analyze only one of the two cases due to symmetry, and let us take the first one, i.e.,

$$(x^k, y^k, z^k) = \left(t(a^2 - b^2), 2abt, t(a^2 + b^2)\right) \quad \text{(K.7).}$$

According to equality (K.7), we have the following equations:

$$x^k = t(a^2 - b^2) \quad \text{(E.1)}$$

$$y^k = 2abt \quad \text{(E.2).}$$

$$z^k = t(a^2 + b^2) \quad \text{(E.3).}$$

The only substitution we can make for these three equations is determining the value of t such that the exponent k is removed. We thus consider three cases:

1st Sub-case:

In this case, we force the equations $(E1, E2, E3)$ to be satisfied by $z^k = t(a^2 + b^2)$ (E.3), so that z is an integer. This implies that $t = (a^2 + b^2)^{k-1}$ (E.4). Therefore, $z = a^2 + b^2$ (E'.3). By substituting (E.4) into (E.2), we obtain:

$$y^k = 2abt = 2\alpha b(a^2 + b^2)^{k-1} \text{ (E'.2)}$$

This implies that:

$2ab = (a^2 + b^2) \iff (a - b)^2 = 0 \iff a = b$

However, from (E.1) we have $x = 0$. Consequently, the only solution is the triad $(x, y, z) = (0, 2ab, 2ab) = (0, 2a^2, 2a^2)$, i.e., $x = 0$ and $y = z$. This contradicts the hypothesis that $x, y, z \neq 0$. Hence, we reject the hypothesis that $t = (a^2 + b^2)^{k-1}$ (E.4).

2-nd Subcase.

Similarly we will require that the equations (E.1, E.2, E.3) with $y^k = t(2ab) \quad (E.2)$ are satisfied by such that y is an integer but we must $t = (2ab)^{k-1} \quad (E.5)$

Therefore it will be $y = (2ab)(E'.2)$. By substituting relation (E.5) into (E.3) we obtain that:

$$z^k = t(a^2+b^2) = (2ab)^{k-1}(a^2+b^2) \quad (E'.3)$$

which means that it must hold

$$2ab = (a^2+b^2) \iff (a-b)^2 = 0 \iff a = b.$$

but this implies from (E.1) that $x = 0$. Thus it will ultimately hold for the triad $(x,y,z) = (0,0,2ab,2ab) = (0,2a^2,2a^2)$ i.e., $x = 0, y = z$ which contradicts the hypothesis $x \cdot y \cdot z \neq 0$. So the hypothesis that we accept that $t = (2ab)^{k-1}$ (E.5).

3rd Subcase.

Similarly we will require that the equations (E.1, E.2, E.3) with $x^k = t(a^2-b^2) \quad (E.1)$ are satisfied by such that x is an integer, so that it must be $t = (a^2-b^2)^{k-1} \quad (E.6)$. Subsequently it will be:

$$x = (a^2-b^2) \quad (E'.1).$$

Substituting (E.6) into (E.3) will give that:

$$z^k = t(a^2+b^2) = (a^2-b^2)^{k-1}(a^2+b^2) \quad (E''.3).$$

which implies that it must hold:

$$a^2-b^2 = a^2+b^2 \iff (2b) = 0 \iff b = 0$$

Thus it will eventually hold for the triad $(x,y,z) = (2ab,0,2ab) = (2a^2,0,2a^2)$, i.e., $y = 0, x = z$ which contradicts the hypothesis $x \cdot y \cdot z \neq 0$. So the hypothesis is rejected to accept that $t = (a^2-b^2)^{k-1}$ (E.6).

Therefore we conclude that Fermat's equation when $n > 2$ positive even integer has no integer solutions for the triad (x,y,z).

II.) The case when $n = 2k+1, k \in \mathbb{Z}^+_{\geq 1}$

For the Fermat equation to hold with $n = 2k+1, k \in \mathbb{Z}^+_{\geq 1}$ we must have the equality

$$x^n + y^n = z^n \iff x^{2k+1} + y^{2k+1} = z^{2k+1} \iff \left(\frac{z}{y}\right)^{2k+1} + \left(\frac{x}{y}\right)^{2k+1} = 1 \quad (F.1)$$

$x.y.z \neq 0, k \in \mathbb{Z}^+_{\geq 1}$. But we do not know which form of inequality the immediately preceding exponent will satisfy for Fermat's equation, i.e., if $n = 2k+1, k \in \mathbb{Z}^+_{\geq 2}$ which inequality will hold for the equation of Fermat if $n = 2k, k \in \mathbb{Z}^+_{\geq 2}$. For our generalized case we will have to prove that the following system will generally hold if $k \in \mathbb{Z}^+_{\geq 2}$:

$$\begin{cases} \left(\frac{z}{y}\right)^{2k+1} - \left(\frac{x}{y}\right)^{2k+1} = 1 \quad (F.1) \\ \left(\frac{z}{y}\right)^{2k} - \left(\frac{x}{y}\right)^{2k} < 1 \quad (F.2) \\ x \cdot y \cdot z \neq 0 \end{cases}$$

For this proof we will use simple elementary functional analysis and to simplify the relations

$$\frac{z}{y} = m, \quad \frac{x}{y} = s,$$

we will set the substitutions $m, s > 0$. Therefore we come to find integer positive solutions for the system

$$\left.\begin{cases} (m)^{2k+1} - (s)^{2k+1} = 1 \quad (F'.1) \\ (m)^{2k} - (s)^{2k} < 1 \quad (F'.2) \\ m, s \neq 0 \end{cases}\right) \iff (1+s^{2k+1})^{2k} < (1+s^{2k})^{2k+1} \quad (F.3)$$

Our problem ultimately comes down to proving that

$(1 + s^{2k+1})^{2k} < (1 + s^{2k})^{2k+1}$ (F.3)

$s > 0$

In general we do not exclude the case $x = 0$, i.e., $s = 0$ as a solution, not of course acceptable but possible. To show what value the variable takes in equation (F.3) we can develop the relation according to a) With Newton's binomial and (F.3) is transformed into the following form

$$(1 + s^{2k+1})^{2k} - (1 + s^{2k})^{2k+1} =$$

$$\sum_{r=1}^{2k-1} s^{2k(2k-r+1)-r} \frac{2k!}{r!(2k-r)!} \left(1 - \frac{2k+1}{2k-r+1} s^r\right) - s^{2k}(2k+1) < 0 \quad (F.4)$$

The less (<) than zero follows because it must

$$1 - \frac{2k+1}{2k-r+1} s^r < 0$$

which is true because

$$\frac{2k+1}{2k-r+1} > 1$$

always, and is also true especially when $s \geq 1, r \geq 1$ therefore $s^r > 1$. Consequences of the above for the function with respect to s will be proportional to the sign of

$$F(s) = (1 + s^{2k+1})^{2k} - (1 + s^{2k})^{2k+1} < 0 \, \forall s > 0$$

b) Similarly, using the limit, we can prove equation $(F.3)$ but more generally for all values of s in R, by the following calculations

For $s \in \mathbb{R} \setminus \{0\}$, $\log(s) > 0$, $k > 1$, from $(F.3)$ if $m = 2k$, $m \in \mathbb{Z^+}_{\geq}4$, we will have the following limit which is obvious:

$$\lim_{m \to +\infty} \frac{1 + s^{m+1}}{1 + s^m} = \lim_{m \to +\infty} \frac{1 + s^{m+1}}{s^m} = \lim_{m \to +\infty} s^{-m}(1 + s^{m+1}) = s, log(s) > 0$$

From $\lim_{m \to +\infty} \frac{(1+s^{m+1})}{(1+s^m)} = s$ we can transform this limit into the inequality:

$$\frac{(1 + s^{m+1})^m}{(1 + s^m)^m} \leq s^m \iff \frac{(1 + s^{m+1})^m}{(1 + s^m)^m} < 1 + s^m \iff (1 + s^{m+1})^m < (1 + s^m)^{m+1}$$

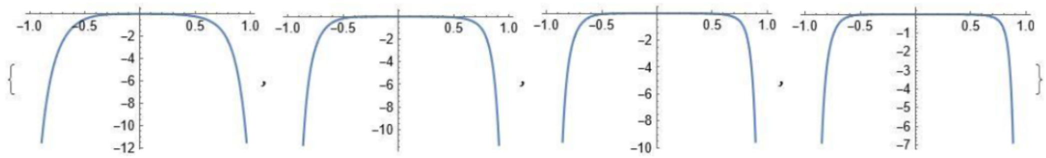
Therefore,

$$F(s) = (1 + s^{2k+1})^{2k} - (1 + s^{2k})^{2k+1} < 0 \, \forall s > 0$$

and $F(s) = (1 + s^{2k+1})^{2k} - (1 + s^{2k})^{2k+1} > 0 \, \forall s < 0$.

So we have proven up to this point that for $s > 0$, $F(s) < 0$, and for $s < 0$, then $F(s) < 0$ as well, with the only root being $s = 0$, as easily proved, when $F(s) = 0$. This means that it is impossible for $F(s)$ to have any roots for s in $\mathbb{R}$ other than zero.

c) All that remains to be absolutely sure is to find where the first derivative of F becomes zero and in which interval it is strictly increasing and in which interval it is strictly decreasing. In this way we will have isolated in $\mathbb{R}$, and of course in the set of positive integers the root s so that will apply $F(s) = 0$ According to the elementary analysis of functions, since the function is negative in the interval $[0, \infty)$, then the function $F'(s) < 0$ and therefore it will be decreasing, as shown in Figure 1. This implies that it has an extremum which is a maximum. However, for the interval $(-\infty, 0]$, in sequence it will be strictly increasing and $F'(s) > 0$. This behavior follows from equation (F.4). To clarify exactly what the value of n is and its properties, we have to calculate the first and second derivatives.



k=2, 2k+1=5 k=3, 2k+1=7 k=4, 2k+1=9 k=5, 2k+1=11

i) The first derivative shall be calculated as follows
The first derivative $F'(s)$ with respect to s is given by:

$$F'(s) = \frac{\partial}{\partial s}\left[(1+s^{2k+1})^{2k} - (1+s^{2k})^{2k+1}\right]$$

Using the chain rule, we can develop this derivative as follows:

$$F'(s) = 2k\cdot(1+s^{2k+1})^{2k-1}\cdot\frac{\partial}{\partial s}(1+s^{2k+1}) - (2k+1)\cdot(1+s^{2k})^{2k}\cdot\frac{\partial}{\partial s}(1+s^{2k})$$

Simplifying and substituting these results back into the expression we obtain

$$F'(s) = 2k(2k+1)s^{2k-1}\left[s(1+s^{2k+1})^{2k-1} - (1+s^{2k})^{2k}\right] \qquad (F.5)$$

i.1) For $s > 0$ and $k > 1$ we will have the following relations:

$$F'(s) = -2k(1+2k)s^{2k-1}\left(1+s^{2k}\right)^{2k} + 2k(1+2k)s^{2k}\left(1+s^{1+2k}\right)^{2k-1}$$

$$2k(2k+1)s^{2k}((1+s^{1+2k})^{2k-1} - \frac{(1+s^{2k})^{2k}}{s})$$

We have the final relation for $F'(s)$:

$$F'(s) = 2k(2k+1)s^{2k}\left((1+s^{1+2k})^{2k-1} - \frac{(1+s^{2k})^{2k}}{s}\right) \qquad \text{(F'.5)}$$

To determine the sign of $F'(s)$, i.e., whether it is greater than or less than zero, we need to examine $\lim_{s\to+\infty} F'(s)$, then

$$\lim_{s\to+\infty} F'(s) = \lim_{s\to+\infty} 2k(2k+1)s^{2k}\left((1+s^{1+2k})^{2k-1} - \frac{(1+s^{2k})^{2k}}{s}\right)$$

$$= 2k(2k+1)\lim_{s\to+\infty}\left(\frac{s^{2k}(1+s^{2k})^{2k}}{s}\right)\cdot\lim_{s\to+\infty}\left(-1+\frac{(1+s^{1+2k})^{2k-1}}{\frac{(1+s^{2k})^{2k}}{s}}\right)$$

$$= 2k(2k+1)\lim_{s\to+\infty}\left(\frac{s^{2k}(1+s^{2k})^{2k}}{s}\right)\cdot\left(-1+\frac{s\lim_{s\to+\infty}\frac{(1+s^{1+2k})^{2k}}{(1+s^{2k})^{2k}}}{(1+s^{2k+1})}\right)$$

$$= 2k(2k+1)\lim_{s\to+\infty}\frac{s^{2k}(1+s^{2k})^{2k}}{s}\cdot\lim_{s\to+\infty}\left(-1+\frac{s}{(1+s^{2k+1})}s^{2k}\right)$$

$$= 2k(2k+1)\lim_{s\to+\infty}\frac{s^{2k}(1+s^{2k})^{2k}}{s}\cdot\lim_{s\to+\infty}\left(\frac{-1}{(1+s^{2k+1})}\right)$$

$$= 2k(2k+1)\lim_{r\to+\infty}\frac{(1+s^{2k})^{2k}}{s^2}\cdot\lim_{s\to+\infty}\left(\frac{-s^{2k+1}}{1+s^{2k+1}}\right)$$

$$= -2k(2k+1)\lim_{s\to+\infty}\left(\left(\frac{s^{2k}+1}{s^{1/k}}\right)^{2k}\right) = -2k(2k+1)\lim_{s\to+\infty}(s^{4k^2-2})$$

$$= -\infty, \quad k\in\mathbb{Z}^+{}_{>1}.$$

We see clearly now that $F'(s) < 0$ for all values of $s\in\mathbb{R}^+, \log(s) > 0$, and therefore the function is decreasing in the interval $s\in(0,+\infty)$.

i.2) $s < 0$ and $k > 1$ also we will have the following relations

$$\lim_{r\to+\infty}F'(r) = \lim_{r\to+\infty}\left[2k(2k+1)r^{2k}\left((1-r^{1+2k})^{2k-1} - \frac{(1+r^{2k})^{2k}}{-r}\right)\right]$$

$$= 2k(2k+1)\lim_{r\to+\infty}\left[r^{2k}\cdot\frac{(1+r^{2k})^{2k}}{-r}\cdot\lim_{r\to+\infty} -1 + \frac{(1+r^{1+2k})^{2k-1}}{\frac{(1-r^{2k})^{2k}}{-r}}\right]$$

$$= 2k(2k+1)\lim_{r\to+\infty}\left[r^{2k}\cdot\frac{(1+r^{2k})^{2k}}{-r}\cdot\lim_{r\to+\infty}(-1+\frac{-r(1-r^{1+2k})^{2k}}{(1-r^{2k+1})(1+r^{2k})^{2k}})\right]$$

$$= 2k(2k+1)\lim_{r\to+\infty}\left[r^{2k}\cdot\frac{(1+r^{2k})^{2k}}{-r}\cdot(-1+\frac{-r\lim_{r\to+\infty}(\frac{1-r^{1+2k}}{(1+r^{2k})})^{2k}}{(1-r^{2k+1})})\right]$$

$$= 2k(2k+1)\lim_{r\to+\infty}\left(r^{2k}\cdot\frac{(1+r^{2k})^{2k}}{-r}\right)(-1+\frac{-r}{1-r^{2k+1}}\cdot r^{2k})$$

$$= 2k(2k+1)\lim_{r\to+\infty}\left(r^{2k+1}\cdot\frac{(1+r^{2k})^{2k}}{r^2}\right)(\frac{-1}{1-r^{2k+1}})$$

$$= 2k(2k+1)\lim_{r\to+\infty}\left(\frac{(1+s^{2k})^{2k}}{r^2}\right)\lim_{r\to+\infty}\frac{-r^{2k+1}}{1-r^{2k+1}}$$

$$= 2k(2k+1)\lim_{r\to+\infty}\left(\frac{r^{2k}+1}{r^{1/k}}\right)^{2k}$$

$$= +2k(2k+1)\lim_{r\to+\infty}(r^{4k^2-2}) = +\infty, \quad k\in\mathbb{Z}^+{}_{>}1$$

We also observe once again that for all values of $s=-r$, the derivative $F'(s) > 0$, and therefore strictly increasing function in the interval $s\in(-\infty,0)$. Additionally, within the intervals $(0,+1)$ and $(-1,0)$, the limits of $F(s)$ tend to 0^- and 0^+, respectively and they take the sign where they approach, it is not necessary to examine them for here. If we set the second derivative $F''(s) = 0$, then again we find $s = 0$. This root is the only real one in the interval $(-\infty,\infty)$, but it is not a turning point because we have two hollows for the function in continuous intervals, and because $F''(s) < 0$ (as can be easily proved with values in the two intervals), it will imply a total maximum. Since the function $F(s)$ is strictly decreasing in the interval $(0,\infty)$ where $F'(s) < 0$, and strictly increasing in the interval $(-\infty,0)$ so $F'(s) > 0$, we again conclude that the point $s = 0$ is the total maximum and it is impossible for there to be another point in the domain $\mathbb{R}$ of the function $F(s)$ such that $F(s) = 0$.

(ii) Then the second derivative will be:

$$F''(s) = \frac{\partial}{\partial s}\left[-2k(1+2k)s^{2k-1}(1+s^{2k})^{2k} + 2k(1+2k)s^{2k}(1+s^{1+2k})^{2k-1}\right]$$
$$= -2k(-1+2k)(1+2k)s^{2k-2}(1+s^{2k})^{2k} - 8k^3(1+2k)s^{4k-2}(1+s^{2k})^{2k-1}$$
$$+ 2k(-1+2k)(1+2k)^2s^{4k}(1+s^{2k+1})^{2k-2} + 4k^2(1+2k)s^{2k-1}(1+s^{1+2k})^{2k-1}$$

If we set as zero the second derivative $F''(s) \leq 0$, then again $s = 0$. This root will be the only real one in the interval $(-\infty, \infty)$, but it is not the inflection point because we have two concave for the function in continuous intervals because $F''(s) < 0$, indicating a total maximum. Since the function $F(s)$ is strictly decreasing in the

interval $[0, \infty)$, i.e., $x_1 < x_2 \implies F(x_1) > F(x_2)$ and therefore $F'(s) < 0$ also and since it is strictly increasing in the interval $(-\infty, 0)$ then $F'(s) > 0$, we conclude again that the point 0 is the total maximum.

From the above, it follows that the only acceptable solution for the data of our system will be $s = 0$, which implies that $x = 0$. But this solution is rejected because we accept it as a hypothesis $x \cdot y \cdot z \neq 0$. But the latter means that the system:

$$\begin{pmatrix} \left(\frac{z}{y}\right)^{2k+1} - \left(\frac{x}{y}\right)^{2k+1} = 1 \quad \text{(F.1)} \\ \left(\frac{z}{y}\right)^{2k} - \left(\frac{x}{y}\right)^{2k} < 1 \quad \text{(F.2)} \\ x \cdot y \cdot z \neq 0 \end{pmatrix}$$

$$\iff$$

$$\begin{pmatrix} \left(\frac{z}{y}\right)^{2k+1} - \left(\frac{x}{y}\right)^{2k+1} = 1 \quad \text{(F.1)} \\ \left(\frac{z}{y}\right)^{2k} - 0 = 1 \quad \text{(F.2)} \\ x \cdot y \cdot z \neq 0 \quad \text{and} \quad x = 0 \end{pmatrix}$$

Which causes a contradiction because there is no integer solution for x, except the zero solution, and therefore for the exponent $n = 2k + 1$, $k \in \mathbb{Z}^{+}{}_{\geq 2}$, there is no integer solution because of the rejection of any zero solution by the hypothesis $x \cdot y \cdot z \neq 0$. It has also been proved (18) that Fermat's equation for $n = 3$ and $n = 4$, has no integer solution, and thus we include the more general case $n = 2k + 1$ or $n = 2k$, $k \in \mathbb{Z}^{+}{}_{>1}$. Therefore for $n > 2$, the Diophantine equation:

$$x^n + y^n = z^n$$

has no integer solutions (x, y, z) with $x \cdot y \cdot z \neq 0$, and Fermat's Last Theorem is proved.

VIII. Indicative values of the allowed values of n, in relation to the number of variables d in the equation 2 parts of theorem

Using the same method, we categorize using logical inequalities to obtain a comprehensive view of both acceptable and rejected values of n. This approach avoids addressing each case individually, which is considered time-consuming and challenging. In each instance, we utilize the relationships $\Theta'.9$ and $\Theta'.10$. For each combination of s and d of interest, we determine the minimum value from $\Theta.9$ as defined in Theorem 5 (click here to refer to Theorem 5).

$$\epsilon_{\min} = \frac{\log_2(d - 2s - 1)}{\log_2\left(\frac{d+1}{d}\right)}, \quad d \in \mathbb{Z}^{+}_{>3}.$$

If we consider the case $d = 15$, the values of n_{down} and n_{up} for $s = 1, 2, 3, 4, 5, 6$ are determined according to the relation

$$d - 2s - 1 > 1 \iff s < \frac{d-1}{2} \quad \text{and we have the final form of } s \text{ as } s \leq \frac{d-3}{2}.$$

This relation determines how the Diophantine equation $(\Theta'11)$ is divided into two parts: $(d-s-1)$ to the left and $s+1$ to the right, as shown in Table 2. Additionally, from the same table, we can see for which values the exponent of n has a solution, ranging from $n_{\text{down}} = \lfloor \epsilon_{\min} \rfloor$ to $n_{\text{down}} \geq 1$, depending on the values of s.

s	$d-s-1$	$s+1$	$\epsilon_{\min}$	n_{down}	n_{up}
1	13	2	38.50269	38	39
2	12	3	35.67769	35	36
3	11	4	32.22016	32	33
4	10	5	27.76264	27	28
5	9	6	21.48011	21	22
6	8	7	10.74005	10	11

Table 2: The values for n_{down}, n_{up} for the 2-segment Diophantine format (Θ'11)

The values of n_{down} , n_{up} if s = 1,2,3, ,5,6, determine when the bipartite Diophantine equation has a solution and when it does not for the specific value d = 15 as an example. We see then for the values of n_{down} what are the limits of the exponent n that the Diophantine equation Θ'11 can be solvable and this is a theoretical strict criterion, which follows from the specific inequalities.

Conclusion

Theorems 1,2,3, and 7, solve, in general, the problem of how many solutions exist or do not exist for Diophantine equations of degree n with d variables. Our goal is not to find such solutions in particular but to prove whether or not solutions exist. Thus, we use this method to find the bounds that n must strictly satisfy with respect to the number of variables d, so that Diophantine equations of such forms have solutions. It is the most modern and fastest proof with mathematical logic conditions.

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Foot notes

De Alwis, A. C. W. L. Solutions to Beal's Conjecture, Fermat's Last Theorem and Riemann Hypothesis.