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Structures Of Pythagorean Triples And Fermat's Last Theorem

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Pythagorean triples are groups of three natural numbers and are a simplification of the notation. They satisfy the Pythagorean theorem when each number is squared and the sum of the first two numbers equals the square of the third number. For example, the triple (3, 4, 5) represents $3^2 + 4^2 = 5^2$. Fermat`s last theorem states that there are no natural numbers for x, y, z, n in this general form $x^n + y^n = z^n$, where is n greater than 2.

The following article shows all the arithmetic relationships that can be found in the Pythagorean triples. Structures in these triples and calculation methods are presented. It also explains how to classify them and how to create infinitely many new Pythagorean triples in these classes. To obtain new classes of Pythagorean triples, is to create with one number the first triple of a new class and continue infinitely in this class by knowing the universal arithmetic structures and relations.

Arithmetic methods for trigonometric problems were maybe applied thousands of years ago. A spectacular find of the Pythagorean triples is around 3700 years old and dates back to Babylonian times [1]. We can therefore assume that the Pythagorean triples were well known and calculable at that time. The identifiable simple artithmetic structures assist in the calculation, classification and creation of the Pytaghorean triple. Perhaps the knowledge was lost at some point.

Geometric and arithmetic solutions illustrate the Phythagorean triple can be traced back to a single value. A core value dependency, is shown with the angle functions and arithmetic interconnections. Restrictions for multidimensional (exponent >2) additions thus become indirectly apparent. The conditions for Fermat's theorem are known due to their connection with the binomial formulas, are shown by algebraic relationships and can also be seen in relation to the unit circle.

When following analyzed the Phythagorean triple, it is important to consider the position of the numbers. The positions of the numbers in the triples are described following as equivalent to the Pythagoras` theorem: cathethus 1 for the first number, cathethus 2 for the second number and hypotenuse for the third number. Geometrically seen, cathethus 2 results from the angle of ascent and arithmetically, it is the smallest number in the triple.

Relevance for the Last Theorem of Fermat

The Pythagoras` theorem is well known and because of the famous Fermat last theorem there are exists several ways for solution and explanations. One Explanation in Table 1 is to find in literature [2] and is working with two quadrats. The larger square is created from the cathets of 4 triangles, which together with their hypotenuses form a smaller square. In explanation 2, which I have developed analogue, the triangles are directed inwards. What remains is a small square that results from the square of the difference between the two cathets.

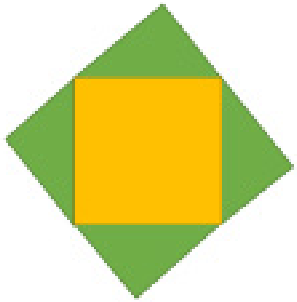
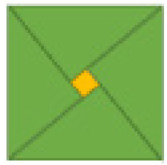
Solutions for the theorem of Pythagoras	Explanation 1	Explanation 2
Conditions: a,b ∈ N, a>b	$a^2 + b^2 = (a+b)^2 - 2ab$	$a^2 + b^2 = 2ab + (a-b)^2$
Geometrical representing		

Table 1: Two Solutions for the Theorem of the Pythagoras

The close connection between Pythagorean theorem and the binomial formulae is remarkable. The arithmetic and geometric solutions illustrate the fundamental interdependence, see Table 1. One single term completes the binomial formula for quadratic exponents and makes it possible to calculate the solution for the Pythagorean theorem as well. The lengths of the cathets in the Pythagorean theorem can be determined from binomial formulae directly using this quadratic extension, refer to (1), where a, b ∈ N, a>b:

$$(a+b)^2 - (a-b)^2 = 4ab \qquad (1)$$

These solutions are available for two-dimensional summations. Exponents greater than two require multidimensional terms, which must also be mixed-quadratic terms. More than one term is therefore necessary, see Pascal's triangle. Simple calculations are no longer suitable due to the multidimensionality and the increasing number of supplementary terms. Calculations with two-dimensional, integer summands can also be traced back to only one variable. It is known from observing the relationships in the unit circle:

$$x^2 + y^2 = r^2 \qquad x,y,r \in N \qquad (2)$$

$$(x - m_1)^2 + (y - m_2)^2 = r^2 \qquad x,y,m,r \in N \qquad (3)$$

with: $x = r \cos \alpha + m_1$ and $y = r \sin \alpha + m_2$ (4)

$$(r \cos \alpha)^2 + (r \sin \alpha)^2 = r^2 \qquad (5)$$

Equation 5 therefore shows the dependency of the entire equation on one angle. If the angle relationships are replaced by concrete values, the dependency of the entire equation on just one value is shown:

for $\alpha = 60t$: $\quad (r\frac{1}{2})^2 + (r\frac{\sqrt{3}}{2})^2 = r^2 \qquad r \in N \qquad (6)$

This results in $\quad \frac{1}{4}r^2 + \frac{3}{4}r^2 = r^2 \qquad (7)$

and also in: $\quad r^2 + 3r^2 = 4r^2 \qquad (8)$

Equations 5 and 8 show that all numbers of the Pythagorean triple can be traced back to one value, i.e. starting from a suitable angle or a number, all quadratic equations can be formed. To form quadratic equations with natural numbers, the inherent relationships must be taken into account.

With quadratic exponents (two-dimensional summations), an infinite number of solutions can be obtained using simple additive and multiplicative operations. However, this is still done by testing. A calculation method is shown below.

Arithmetical Calculation of Pythagorean Triples

Identify the Structure of the Triple

By analysing the structure of the triples, following are identified the classes they contained. Generating from one number is develop the first triple in a class, which has a typical structure. It is described the methods that make it possible to obtain an infinite number of further triples in each class. Universal rules for the generation of first triple in a class and develop further triple in this class are given. The classes differ according to two aspects.

- Is cathetus 2 an even or uneven number?
- Which difference is between hypothenuse and cathetus 2?

Triple of Even and Uneven Numbers– Any Number can be used

For even values of cathetus 2, the difference between hypothenuse and cathetus 1 is 2 and for uneven values of cathetus 2, the difference is 1. This is important for the formation of new classes of Pythagorean triples. The distance from the cathetus 2 to the hypotenuse determines the class of the triple and is characteristic and essential for arithmetic calculation rules.

Subdivision according to the number of the first cathetus	For even values for the cathetus 2:	For uneven values for the cathetus 2:
Value of cathetus 2	$K_2 = 2n$	$K_2 = 2n+1$
Difference between hypotenuse and cathetus 2 ($H-K_2$)	2	1
Triple class	2	1
Example for triple of this class	3,4,5 or 8,6,10	4,3,5
Further examples	15,8,17 24,10,26	12,5,13 24,7,25
	35,12,37 …	40,9,41 …

Table 2: Example of the Simplest subdivision of Triple Classes

The right-angled triangle that can be represented with it is the one with the smallest rise in this class. In these simple cases in table 2, the differnces of cathetus 1 and hypotenuse only differ by the value 1 or 2. They can be differentiated depending on the even or uneven cathetus 2. This trivial cases – always the first triple in a class - are triples that are geometrically maximally compressed triangles, see figure 1.

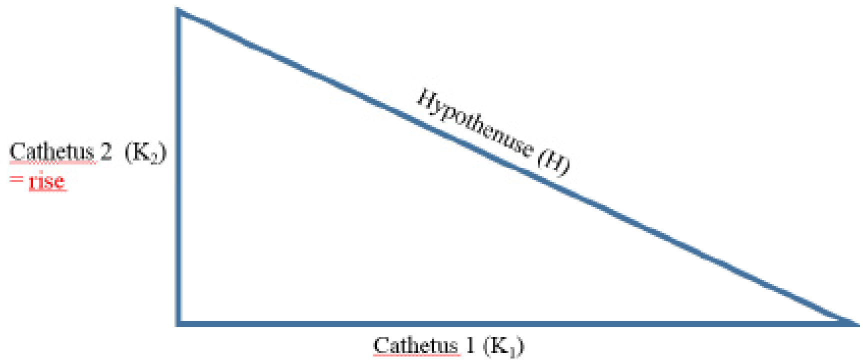


Figure 1: Representation of the Pythagorean triple in the Triangle

Formation of New Classes of Pythagorean Triples

The properties of the number of cathetus 2 (3 or more, even or uneven) determine the calculation rules of the first triple in a class. The starting point is a square number or twice a square number of the cathetus 2. Due to the formation rule, the sum of the other two sides is a multiple of the side that forms the slope. To consider the position of each number is the base for identifying the class of triple and allows further calculations of triple in this class. The exact procedure can be seen with examples for both variants in table 3.

For even square numbers, the summands into which the double square number is divided must also differ by the difference 2. Splitting results in the values for cathetus 1 and the corresponding hypotenuse of the first triple of a class. In my example, the assumed even number is 10, squaring it makes it 100 and then doubling it makes it 200. Using the example of the even square number, the value for cathetus 2 is 10, twice the assumed square number. As a result, this triple of the even and doubled square number 200: 99, 20, 101 is the first in its class.

Triple formation rules for ($H-K_2)>2$	For even values for the cathetus 2:	For uneven values for the cathetus 2:
Condition for selection of a number	Double of a square number	Square number
Examples	200	121
Calculation K_2	$K_2 = 2\sqrt{\frac{200}{2}}$	$K_2 = \sqrt{121} = 11$
Value of K_2 in Examples	20	11
Decomposition into summands, whereby the values for K_1 and H are formed	200 = 99 + 101	121 = 60 + 61
Examples of first triples	99, 20, 101	60, 11, 61
General rules for each first triples in a class ($a,b,c,n \in N$)	For: $2b^2 = 2n$ $a + c = 2b^2$ $c = a + 2$	For: $b^2 = 2n+1$ $a + c = b^2$ $c = a + 1$

Table 3: Examples for Formation of the First Pythagorean Triple of a Class

For example 121, as the square number of 11. Cathete 2 is the assigned number 11. Then the square number is divided into two summands. After the division into the two addends, they may only differ by the value 1 in the case of an odd square number. This results in the first triple of this class for the example of the odd square number 121: 60, 11, 61.

Pythagorean triple of the higher classes ($H-K_2)>2$	Even values for the cathetus 2:	Uneven values for the cathetus 2:
Examples of first triples	35, 12, 37 99, 20, 101 143, 24, 145 195, 28, 197 255, 32, 257	40, 9, 41 60, 11, 61 84, 13, 85 112, 15, 113 180, 19, 181
	323, 36, 325 ...	220, 21, 221 ...
Difference between H and K_1	2	1

Table 4: The First Pythagorean Triple of a Class

First triples of a class form the basic pattern for all further triples in this class. Important is the property of the value of cathetus 2, is to see as core number. For each first triple in a class, the difference value 1 or 2 between the cathetus 1 and the corresponding hypotenuse is significant. Based on the first triple of a class formed, it is possible to calculate the subsequent triples within a class using simple arithmetic rules.

Triples of Higher Classes – the Options are Reducing

Higher triples of a class differ by larger differences between cathetus 1 and hypotenuse. The arithmetic relationships are astonishing, allowing suitable relations to be found, which can be used to form further triples according to simple rules. Difference between hypotenuse and cathetus 2 contains a lot of information. They determines the class of triples. A "gradient factor x" can be calculated from these differences. The way in which the gradient factor is determined depends on whether the value of cathetus 2 is even or uneven.

Universal Calculation Methods for Pythagorean Triples in a Class

The calculation rules are based again on the property of cathetus 2 to be an even or uneven number. The next triple is obtained via universal formation rules that can always be used in the same way for the two cases, see table 3. The difference value of cathetus 2 and hypotenuse is the key and determines the value of the consecutive cathetus 1 and forms a constant ratio. With the help of the determined gradient factor and the knowledge of its mathematical relationships, a subsequent triple can be calculated.

Once the pattern of a triple equation has been identified, it is also possible to identify all subsequent triples in this class. By knowing the structure, no iterative calculation is necessary there, but can be carried out at any point in the number range, taking into account the constant differences for the values of the consecutive cathetus 1.

The following arithmetic relationships contain all the steps for forming any triple of a class. The gradient factor x can be read from the first triple of each class. It forms the basis for all necessary calculations.

Examples	For even values for the cathetus 2:	For uneven values for the cathetus 2:
Triple	15, 8, 17	24, 7, 25
Difference between value for hypotenuse and cathetus 2 - general:	$H\text{-}K_2 = x^2$	$H\text{-}K_2 = 2\cdot x^2$
Difference ($H\text{-}K_2$)	9	18
The gradient factor x - Decomposition of the difference between hypotenuse and cathetus 2 value – general:	x^2	$2\cdot x^2$
The gradient factor x - Decomposition of the difference between hypotenuse and cathetus 2 value	3^2	$2\cdot 3^2$
The gradient factor x	3	3
Calculation of the cathete 1 value for the following triple $K_{n1} \succ K_1$ (K_{n1} - consecutive cathetus 1) – general:	$K_{n1}\text{-}K_1 = 2\cdot x$	$K_{n1}\text{-}K_1 = 2\cdot 2\cdot x$
Calculation of the cathetus 1 value from the following triple	$2\cdot 3$	$2\cdot 2\cdot 3$
Calculated difference between two consecutive catheters 1 ($K_{n1}\text{-}K_1$), $K_{n1} \succ K_1$	6	12
Next value for K_1	21	36
Calculation of K_2 - general:	$K_2 = (K_1{}^2 /(H\text{-}K_2) – (H\text{-}K_2)/2$	

Calculation of K_2-general for this triple group	$(K_1^2 /\mathbf{9} -\mathbf{9}) /2$	$(K_1^2 /\mathbf{18} -\mathbf{18}) /2$
Next value for K_2	20	27
Next value for H	29	45
Next triple formed	21, 20, 29	36, 27, 45

Table 5: Determine with Gradient Factor x The Values for the Following Triple

To form the following triple in a class, the value of cathetus 2, even or uneven, and the difference between the values of the hypothenuse (H) and cathetus 2 (K_2) is required for calculation rules. The difference ($H-K_2$) defines the classes of Pythagorean triples and is constant and the type of value of cathetus 2, even or uneven, detemine the calculation steps.

In this way, all 'working' combinations for the Pythagorean triples can be generated. Thus there is a universal formation rule for all Pythagorean triples.

Simple rules allow their formation and the development of further triples from the initial triples – first formable triple in a class -, taking their properties into account. It is important to determine the gradient factor x, which defines a triple class. With this gradient factor and the simple arithmetic relationships, it is possible to form an infinite number of triples. The varied use of the number 2 as a factor and exponent is remarkable and gives the arithmetical relationships an almost trivial appearance.

Summary

Starting with one number can be formed a Pythagorean triple, which has defined properties and is characterized in a class. In this class it is possible to calculate any number of further triples starting from these first formed triple. Simple calculation steps and patterns show the formation and classification of triples and the infinitely possible calculations of triples in a defined class. For the Pythagorean triples, the possible solutions with natural numbers are trivial to calculate and easy to analyse.

The close interconnection between Pythagoras' theorem and Fermat's last theorem is demonstrated. For more than two-dimensional additions (multi-dimensional), several and also mixed-square terms must be used. For multidimensional calculations, there is no core value dependency as found here in two-dimensional equations. The main aspect is the traceability of every quadratic equation to one value, the strong relationship between a number and there rules for satisfying Phythagora`s theorem.

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