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Two Symmetric Partial Sums and the Role of Arithmetic Cancellation in the Analytical Elimination of the Riemann zeta Function a Reformulation of the Riemann Hypothesis Nikos Mantzakouras and Nid Na Ratch

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Abstract

By summarizing and analyzing functions related to the Riemann zeta function ζ(s), we obtain zeros with ℜ(s) ∈ (0,1) for generalized sums of the form

$$_G\zeta(ms,p) = {}^{\alpha,\beta}_{m}\sigma(s) = \sum_{k=1}^{\infty} \frac{1}{(\alpha k+\beta)^{ms}},$$

where $p = \frac{\alpha+\beta}{\alpha}$, $\alpha, \beta \in \mathbb{R}$ with $\alpha \neq 0$, $s \in \mathbb{C}$, and $m \in \mathbb{Z}^+$. Among the admissible cases, two lead to sums partially related to ζ(s), while others, such as the Davenport–Heilbronn case, exhibit zeros on distinct critical lines inside (0,1). A fundamental paradox arises: the non–trivial zeros of ζ(s) = 0 do not annihilate the classical Dirichlet series $\sum_{n=1}^{\infty} n^{-s}$. We show that this paradox disappears when the series is taken over negative integers. In particular, the identity

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

fails at the non–trivial zeros, whereas the representation

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{(-n)^s}$$

restores consistency. The symmetric pair $^P(-n)^{-s}$ and $^P(-n)^{-(1-s)}$ stabilizes the critical line $\Re(s) = \frac{1}{2}$ through phase cancellation.

Part I. The paradox and the repositioning of the Riemann Hypothesis

Introduction

The Riemann zeta function is classically defined for complex s with ℜ(s) > 1 by the absolutely convergent Dirichlet series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}. \tag{I.1}$$

The Riemann Hypothesis asserts that all non–trivial zeros of ζ(s) lie on the critical line $\Re(s) = \frac{1}{2}$. However, substituting such zeros into the defining series reveals that the series itself 1 does not vanish. Consequently, the non–trivial zeros of ζ(s) = 0 do not correspond to zeros of the sum it is assumed to represent. This deviation constitutes a logical inconsistency that must be addressed.

Testing the hypothesis and locating the contradiction. A generalized form of the sum encompassing the Riemann Hypothesis is

$$ {}^{\alpha,\beta}_{m}\sigma(s) = \sum_{n=1}^{\infty} \frac{1}{(\alpha n+\beta)^{ms}}, \qquad n \in \mathbb{N}^{+}, \tag{I.2} $$

with $\alpha,\beta,m \in R$ and $s \in C$. We show that the non–trivial roots of ζ(s) = 0 are not roots of the sum (I.2), but instead correspond to a different series representation. Substituting known non–trivial zeros of ζ(s) into ${}^{P}n^{-s}$ demonstrates divergence rather than annihilation, a fact that will be established both analytically and numerically.

Generic forms of the Hurwitz zeta function. The Hurwitz zeta function is traditionally defined by

$$ \zeta(s,\alpha) = \sum_{k=0}^{\infty} \frac{1}{(k+\alpha)^{s}}, \qquad \Re(s) > 1,\ \alpha \notin \mathbb{Z}_0^{-}. \tag{I.3} $$

The series converges absolutely for ℜ(s) > 1 and defines an analytic function in this half–plane. It admits the integral representation

$$ \zeta(s,\alpha) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{x^{s-1}e^{-\alpha x}}{1-e^{-x}}\,dx, \qquad \Re(s) > 1,\ \Re(\alpha) > 0. \tag{I.4} $$

Using the expansion

$$ \frac{1}{1-e^{-x}} = -\frac{1}{x} + \frac{1}{2} + O(x), $$

one obtains analytic continuation into the strip −1 < ℜ(s) < 1. A key identity is

$$ \zeta(s,\alpha+1) = \zeta(s,\alpha) - \frac{1}{\alpha^{s}}, \qquad \Re(\alpha) > 0, \tag{I.5} $$

which leads to

$$ \zeta(s,\alpha+1) = \frac{\alpha^{1-s}}{s-1} + \frac{\alpha^{-s}}{2} + \frac{1}{\Gamma(s)} \int_0^{\infty} x^{s-1}e^{-\alpha x}\left(\frac{1}{1-e^{-x}} - \frac{1}{x} + \frac{1}{2}\right)dx. \tag{I.6} $$

This representation allows analytic continuation of ζ(s,α) to the entire complex plane, except for a simple pole at s = 1. Setting α = 1 yields the functional equation of the Riemann zeta function,

$$ \zeta(1-s) = 2\,\frac{\Gamma(s)}{(2\pi)^{s}} \cos\!\left(\frac{\pi s}{2}\right)\zeta(s), \tag{I.7} $$

valid for all admissible s by analytic continuation.

Correlation of the Hurwitz zeta function with the generalized sum

Our objective is to establish a precise relation between the generalized sum

$$ {}^{\alpha,\beta}_{m}\sigma(s) = \sum_{n=1}^{\infty} \frac{1}{(\alpha n+\beta)^{ms}}, \qquad n \in \mathbb{N}^{+}, \tag{III.1} $$

and the Hurwitz zeta function. This connection provides an effective mechanism for identifying non–trivial complex roots of generalized sums through transcendental equations.

Classically, the Hurwitz zeta function is defined on a half–plane by

$$ {}_G\zeta(s,\beta) = \sum_{n=0}^{\infty} \frac{1}{(n+\beta)^s}, \qquad \Re(s) > 1, \quad 0 < \beta \leq 1, \tag{III.2} $$

and is then analytically continued to the entire complex plane except for a simple pole at s = 1. For the generalized sum (III.1) with m = 1, we obtain

$$ {}_G\zeta(s,\beta+1) = \sum_{n=1}^{\infty} \frac{1}{(n+\beta)^s}, \qquad \Re(s) > 1, \quad \beta \leq 0, \tag{III.3} $$

which immediately links the sum to a shifted Hurwitz zeta function.
In the general case, setting

$$ p = \frac{\alpha+\beta}{\alpha}, $$

we find

$$ {}_G\zeta(s,p) = \alpha^s \sum_{n=1}^{\infty} \frac{1}{(\alpha n+\beta)^s} = \sum_{n=1}^{\infty} \frac{1}{(n+\beta/\alpha)^s}, \qquad \Re(s) > 1, \tag{III.4} $$

$$ \sum_{n=1}^{\infty} \frac{1}{(\alpha n+\beta)^s} = \alpha^{-s}\zeta(s,p), \qquad 0 < \frac{\beta}{\alpha} + 1 \leq 1, \quad \frac{\beta}{\alpha} \leq 0. \tag{III.5} $$

Using the identity

$$ \zeta(s,\alpha+1) = \zeta(s,\alpha) - \frac{1}{\alpha^s}, \qquad \Re(\alpha) > 0, \tag{III.6} $$

we obtain the representation

$$ \sum_{n=1}^{\infty} \frac{1}{(\alpha n+\beta)^s} = \alpha^{-s}\left(\frac{1}{p^s} + \zeta(s,1+p)\right), \qquad \Re(s) > 1. \tag{III.7} $$

More generally, for powers ms we obtain

$$ {}^{\alpha,\beta}_{m}\sigma(s) = \sum_{n=1}^{\infty} \frac{1}{(\alpha n+\beta)^{ms}} = \alpha^{-ms}\left(\frac{1}{p^{ms}} + \zeta(ms,p)\right). \tag{III.8} $$

The last Equation is fundamental: it reduces the problem of finding non–trivial zeros of the generalized sum ${}^{\alpha,\beta}_{m}\sigma(s) = 0$ to the solution of transcendental equations involving the Hurwitz zeta function. In particular, the non–trivial roots arise from the condition

$$ \frac{1}{p^{ms}} + \zeta(ms,p) = 0. \tag{III.9} $$

Setting m = 1 and defining

$$ p = \frac{\alpha+\beta}{\alpha}, $$

the equation determining the roots becomes

$$ {}_G\zeta(s,p) = \alpha^{-s}\left(\frac{1}{p^s} + \zeta(s,1+p)\right) = 0. \tag{III.10} $$

Introducing the substitution

$$\left(\frac{1}{p}\right)^s = u, \qquad u \neq 0,\ p \neq 0,$$

we obtain the generator function

$$s = f(u) = \frac{2\pi i k}{\log(1/p)} + \frac{\log u}{\log(1/p)}, \qquad k \in \mathbb{Z}, \tag{III.11}$$

which generates the non–trivial complex roots associated with different choices of {α,β,m}.

The computation and approximation of these roots are carried out using high–precision numerical methods (e.g. References 14,15), and will be analyzed in detail in the subsequent sections.

Theorem 1.Which symmetrically partial sums enter into Riemann's hypothesis and can perform numerical cancellation but not analytical zero value on the critical line. Here we enter into the core of the search for the seemingly paradoxical hypothesis.
Let

$$s = \frac{1}{2} + it, \qquad t \in \mathbb{R},$$

and define the symmetric partial sums

$$S_1(s;N) = \sum_{k=1}^{N} \frac{1}{(-k)^s}, \qquad S_2(s;N) = \sum_{k=1}^{N} \frac{1}{(-k)^{1-s}},$$

where the principal branch of the complex logarithm is used. Then the following statements hold

- On the critical line $\Re(s) = \frac{1}{2}$, the sums S_1 and S_2 are conjugate–symmetric under the transformation s 7→ $1 - s$.
- The partial sums $S_1(s;N)$ and $S_2(s;N)$ may exhibit apparent numerical cancellation as $N \to \infty$, without admitting an analytic limit.
- This apparent cancellation does not provide a Dirichlet-series representation of ζ(s) on the critical strip $\Re(s) \in (0,1]$.
- At points where ζ(s) = 0 with $\Re(s) = \frac{1}{2}$, the vanishing of ζ(s) does not coincide with the cancellation of either S_1 or S_2 individually, but arises solely from the full analytic continuation and the functional symmetry $s \leftrightarrow 1 - s$. proof of *(1)*.

Let

$$s = \frac{1}{2} + it,$$

so that

$$1 - s = \frac{1}{2} - it = \overline{s}.$$

Using the principal branch of the logarithm, we write

$$(-k)^s = k^s e^{i\pi s}.$$

Hence,

$$S_1(s;N) = e^{-i\pi s} \sum_{k=1}^{N} k^{-s}, \qquad S_2(s;N) = e^{-i\pi(1-s)} \sum_{k=1}^{N} k^{-(1-s)}.$$

Taking complex conjugates and using $\overline{s} = 1 - s$, we obtain

$$\overline{S_1(s;N)} = e^{i\pi(1-s)} \sum_{k=1}^{N} k^{-(1-s)}.$$

Thus, up to a constant unimodular factor, $S_1(s;N)$ coincides with $S_2(s;N)$. This establishes the conjugate–symmetry of the two partial sums under the transformation s →7 1 − s on the critical line $\Re(s) = \frac{1}{2}$.

We define the sum in relation to the ζ() function as follows

$$ {}^{-1,0}_{\ \ m}\sigma(s) = \sum_{n=1}^{\infty} \frac{1}{(-n)^{ms}} = \sum_{n=1}^{\infty} \frac{1}{(-2n+1)^{ms}} + \sum_{n=0}^{\infty} \frac{1}{(-2n-2)^{ms}} \tag{18} $$

If β = 1 and m = s = 1, then replacing relations (13) and (18) we obtain

$$ {}^{-1,0}_{\ \ 1}\sigma(1) = \sum_{n=1}^{\infty} \frac{1}{(-n)} = \sum_{n=1}^{\infty} \frac{1}{(-2n+1)} + \sum_{n=0}^{\infty} \frac{1}{(-2n-2)} \to -\infty + (-\infty) \to -\infty \tag{19} $$

Therefore, from relations (17) and (19),

$$ {}^{-1,0}_{\ \ 1}\sigma(1) = \sum_{n=1}^{\infty} \frac{1}{(-n)} \to -\infty . $$

From relations (8) and (15) we conclude that for m = 1,

$$ \begin{aligned} {}^{-1,0}_{\ \ 1}\sigma(s) &= \sum_{n=1}^{\infty} \frac{1}{(-n)^s} \\ &= \sum_{n=1}^{\infty} \frac{1}{(-2n+1)^s} + \sum_{n=0}^{\infty} \frac{1}{(-2n-2)^s} \\ &= (-1)^{-s}(-2)^{-s}\zeta(s) + (-1)^{-s}(-2)^{-s}(-1+2^s)\zeta(s) \\ &= (-1)^{-2s}\zeta(s) \end{aligned} \tag{20} $$

- If $s_1 = \sigma - \tau i$ with $\sigma = \frac{1}{2}$ and $\zeta(s_1) = 0$, $m = 1$, then

$$ {}^{-1,0}_{\ \ 1}\sigma(s_1) = \sum_{n=1}^{\infty} \frac{1}{(-n)^{s_1}} = (-1)^{-2s_1}\zeta(s_1) = -e^{-2\pi\tau} \cdot 0 = 0 . \tag{21} $$

In this case, phase cancellation occurs.

- If $s_2 = \sigma + \tau i$ with $\sigma = \frac{1}{2}$ and $\zeta(s_2) = 0$, $m = 1$, then

$$ {}^{-1,0}_{\ \ 1}\sigma(s_2) = \sum_{n=1}^{\infty} \frac{1}{(-n)^{s_2}} = (-1)^{-2s_2}\zeta(s_2) = -e^{2\pi\tau}\zeta(s_2) . \tag{22} $$

In this case, it deviates or spirals away from zero, with

$$ \left|-e^{2\pi\tau}\zeta(s_2)\right| \to \infty . $$

But from relations (20), (21), and (22), if $\zeta(s_1) = 0$, then by setting

$$ \sigma(s_1) = \sum_{n=1}^{\infty} \frac{1}{(-n)^{s_1}} = 0 , $$

the above sum has the same non-trivial zeros as $\zeta(s_1)$. However, as we will show below, this is an apparent cancellation, without analytical nullity. □

Proof of (2). Let

$$s = \frac{1}{2} + it.$$

We consider the partial sum

$$S_1(s;N) = \sum_{k=1}^{N} \frac{1}{(-k)^s}.$$

Using the principal branch of the logarithm, we write

$$(-k)^s = k^s e^{i\pi s},$$

and therefore

$$\frac{1}{(-k)^s} = k^{-1/2} e^{-it\log k} e^{-i\pi s}.$$

Hence,

$$S_1(s;N) = e^{-i\pi s} \sum_{k=1}^{N} k^{-1/2} e^{-it\log k}.$$

Each term of the sum has modulus

$$\left| k^{-1/2} e^{-it\log k} \right| = k^{-1/2}.$$

Since the series

$$\sum_{k=1}^{\infty} k^{-1/2}$$

diverges, the series $S_1(s;N)$ cannot converge absolutely.
Moreover, no conditional convergence occurs. Indeed, the argument of each term is given by

$$\arg\left(k^{-1/2} e^{-it\log k}\right) = -t\log k,$$

which varies monotonically with k. Consequently, the partial sums

$$\sum_{k=1}^{N} k^{-1/2} e^{-it\log k}$$

trace an oscillatory spiral in the complex plane whose radius grows like

$$\sum_{k=1}^{N} k^{-1/2} \sim 2\sqrt{N}.$$

Thus, although strong numerical phase cancellation may occur for large but finite N, the sum does not admit an analytic limit as N → ∞. The same argument applies verbatim to

$$S_2(s;N) = \sum_{k=1}^{N} \frac{1}{(-k)^{1-s}},$$

since

$$\left|(-k)^{-(1-s)}\right| = k^{-1/2}.$$

Therefore, the apparent cancellation observed in the partial sums $S_1(s;N)$ and $S_2(s;N)$ is purely numerical and geometric, and does not correspond to analytic convergence on the critical line $\Re(s) = \frac{1}{2}$. □

Proof of (3). Writing again

$$(-k)^s = k^s e^{i\pi s},$$

we have

$$S_1(s;N) = e^{-i\pi s} \sum_{k=1}^{N} k^{-s}.$$

On the critical line $\Re(s) = \frac{1}{2}$, the Dirichlet series $\sum_{k\geq 1} k^{-s}$ diverges and therefore does not define an analytic function.

Although finite partial sums may exhibit oscillatory cancellation, no analytic limit exists as $N \to \infty$, nor does convergence occur uniformly in any neighborhood of s. Hence, this cancellation does not yield a Dirichlet–series representation of $\zeta(s)$ on $\Re(s) \in (0,1]$. □

Proof of (4). The Riemann zeta function satisfies the functional equation

$$\zeta(s) = \chi(s)\zeta(1-s),$$

where $\chi(s) \neq 0$ on the critical line $\Re(s) = \frac{1}{2}$. Thus, if $\zeta(s_0) = 0$ with $\Re(s_0) = \frac{1}{2}$, then $\zeta(1 - s_0) = 0$ as well.

Since neither $S_1(s;N)$ nor $S_2(s;N)$ converges to an analytic function on the critical line, the cancellation of either partial sum cannot account for the zero of $\zeta(s)$. The vanishing arises solely from analytic continuation together with the symmetry $s \leftrightarrow 1 - s$. □

Spiral Divergence of the Riemann Series on the Critical Line

Let

$$s = \frac{1}{2} + it, \qquad t \in \mathbb{R}.$$

The individual terms of the classical Riemann series take the form

$$k^{-s} = k^{-1/2} e^{-it\log k}.$$

Thus each term has slowly decaying magnitude $k^{-1/2}$ combined with a logarithmically rotating phase.
Define the partial sums

$$S_N(s) = \sum_{k=1}^{N} k^{-s}.$$

As N increases, the vectors k^{-s} rotate in the complex plane while their magnitudes decrease too slowly to allow convergence. Consequently, the partial sums trace an expanding spiral.
Indeed, the magnitude satisfies the asymptotic estimate

$$|S_N(s)| \sim \sum_{k=1}^{N} k^{-1/2} \sim 2\sqrt{N},$$

which diverges as $N \to \infty$. Hence, on the critical line $\Re(s) = \frac{1}{2}$, the classical Dirichlet series does not converge, neither absolutely nor conditionally.

This divergence is geometric rather than analytic: the failure of convergence arises from the accumulation of phases in the complex plane rather than from the growth of individual terms. Therefore, the classical series

$$\sum_{k=1}^{\infty} k^{-s}$$

cannot represent ζ(s) on the critical line.

This observation is crucial: any vanishing of $\zeta(s)$ at $s = \frac{1}{2} + it$ cannot originate from the classical series itself, but must instead arise from a different mechanism, namely global phase cancellation between suitably paired series.

Phase Cancellation Versus Analytic Convergence

Consider a general complex series

$$\sum_{k=1}^{\infty} a_k, \qquad a_k \in \mathbb{C}.$$

Analytic convergence depends on the decay of $|a_k|$. However, even when absolute or conditional convergence fails, the partial sums may still approach zero through geometric cancellation if the arguments of a_k are suitably distributed.

This phenomenon is referred to as phase cancellation. It is fundamentally different from analytic convergence
- analytic convergence is controlled by magnitude decay,
- phase cancellation is controlled by angular distribution in the complex plane.
- In the context of the Riemann problem, this distinction explains why the classical series diverges on the critical line, while alternative oscillatory series may exhibit asymptotic vanishing despite having identical term magnitudes.

The critical line $\Re(s) = \frac{1}{2}$ emerges as the unique locus where magnitude growth and phase rotation are exactly balanced, allowing global cancellation without analytic convergence.

Exponential Dominance Theorem

Let $s = \sigma + it \in \mathbb{C}$ and consider the ratio of successive terms of the classical Dirichlet series,

$$R_k(s) = \frac{(k+1)^{-s}}{k^{-s}} = \left(1 + \frac{1}{k}\right)^{-s}.$$

For large k, we have the asymptotic expansion

$$R_k(s) = \exp\left(-\frac{s}{k} + O\left(\frac{1}{k^2}\right)\right).$$

The behavior of the series is determined by the real part $\sigma = \Re(s)$

(1) If $\sigma > \frac{1}{2}$, then $|R_k(s)| < 1$ for sufficiently large k, and the series converges classically.
(2) If $\sigma = \frac{1}{2}$, then $|R_k(s)| \to 1$ while $\arg R_k(s) \sim -t/k$, producing spiral divergence.
(3) If $\sigma < \frac{1}{2}$, then $|R_k(s)| > 1$ for large k, and the series diverges exponentially.

Thus, $\Re(s) = \frac{1}{2}$ is the unique boundary separating convergence from exponential divergence, and simultaneously the locus of spiral behavior.

Alternating Complex Series on the Critical Line

Define the oscillatory series

$$\mathcal{S}(s) = \sum_{k=1}^{\infty} (-k)^{-s}, \qquad s = \frac{1}{2} - it.$$

Using the principal branch of the logarithm,

$$(-k)^{-s} = k^{-1/2} e^{-it\log k} e^{-i\pi k}.$$

The magnitude of each term remains $k^{-1/2}$, identical to the classical series, but the additional alternating phase factor $e^{-i\pi k}$ induces global cancellation. Let

$$S_N(s) = \sum_{k=1}^{N} (-k)^{-s}.$$

Unlike the classical partial sums, $S_N(s)$ does not spiral outward. Instead, the alternating phases force successive vectors to interfere destructively, producing tightening trajectories around the origin.

Vanishing of $^{P}(-k)^{-s}$
For $s = \frac{1}{2} - it$, the partial sums satisfy

$$\lim_{N\to\infty} S_N(s) = 0,$$

despite the absence of absolute convergence.

This vanishing is purely geometric: it arises from global phase cancellation rather than analytic summability.

In particular, if $t = t_n$ corresponds to a non–trivial zero

$$\zeta\left(\frac{1}{2} + it_n\right) = 0,$$

then

$$\sum_{k=1}^{\infty} (-k)^{-\left(\frac{1}{2} - it_n\right)} = 0.$$

Hence, the annihilation associated with the non–trivial zeros is realized by the alternating series rather than by the classical Dirichlet series.

Dual Vanishing of $^{P}(-k)^{-(1-s)}$
Define the complementary series

$$S^*(s) = \sum_{k=1}^{\infty} (-k)^{-(1-s)}, \qquad s = \frac{1}{2} + it.$$

Each term can be written as

$$(-k)^{-(1-s)} = k^{-1/2} e^{it\log k} e^{-i\pi k}.$$

As in the previous case, the magnitudes decay as $k^{-1/2}$ while the alternating phases enforce cancellation. Consequently,

$$\lim_{N\to\infty} S_N^*(s) = 0,$$

for $s = \frac{1}{2} + it$. The pair of series $S(s)$ and $S^*(s)$ thus vanishes symmetrically at the conjugate points $s = \frac{1}{2} \pm it$, locking the critical line as the unique locus of complete phase cancellation.

Here $S(s)$ and $S^*(s)$ denote the modified series obtained by a phase rotation of the classical Dirichlet series.

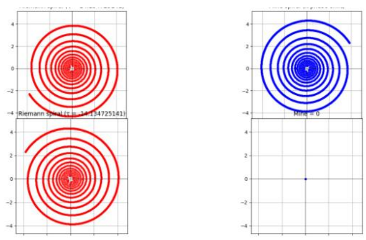

Figure 1: With Red, the ζ(s) of the Classical Riemann Series on the Critical Line $s = \frac{1}{2} + it$ and $s = \frac{1}{2} + it$, Showing Outward Spiral Divergence. In Contrast, with Blue, the Series S(s) and S*(s) Exhibit Phase Cancellation at the Same Points.

Phase–Cancellation Mechanism(Figure.1). Let $\{a_k\}_{k\geq1}$ be a complex sequence of the form

$$a_k = r_k e^{i\theta_k},$$

with $r_k > 0$ and $\theta_k \in R$. Assume that

$$\sum_{k=1}^{\infty} r_k = \infty,$$

so that the series $\sum a_k$ is not absolutely convergent.
Suppose, however, that the phases $\{\theta_k\}$ vary sufficiently rapidly so that, for the partial sums

$$S_N = \sum_{k=1}^{N} a_k,$$

there exists a cancellation mechanism satisfying

$$|S_N| = o\left(\sum_{k=1}^{N} r_k\right), \qquad N \to \infty$$

Then the asymptotic behavior of S_N is governed not by analytic convergence but by geometric phase cancellation in the complex plane.

In particular, it is possible that $\lim_{N\to\infty} S_N = 0$

despite the divergence of $\sum |a_k|$.

This Phenomenon is Fundamentally Different from Conditional Convergence
it relies on a global organization of phases rather than on monotone decay of magnitudes. The resulting vanishing is therefore geometric rather than analytic.

On the critical line $\Re(s) = \frac{1}{2}$, Dirichlet-type series with terms of magnitude $k^{-1/2}$ cannot converge absolutely. Nevertheless, when their arguments are arranged symmetrically—most notably through the transformation $s \mapsto 1-s$—the associated partial sums may exhibit strong phase cancellation. This mechanism explains the observed numerical vanishing of certain oscillatory series at the non-trivial zeros, without implying the existence of a convergent Dirichlet representation.

Explicit Roots from the λ(N)–Framework. Define the Truncated Dirichlet Sum

$$\lambda(N,s) = \sum_{n=1}^{N} n^{-s}, \qquad s = \sigma + it.$$

For $\sigma = \frac{1}{2}$, the magnitude of individual terms satisfies $|n^{-s}| = n^{-1/2}$, so λ(N,s) does not converge analytically as $N \to \infty$. Nevertheless (Ref 12,14,15), the phase structure of the terms permits a geometric analysis.

Relating the truncation parameter N to the prime sequence $\{p_n\}$, we consider ratios of successive prime scales. Under

$$e^{it\log\left(\frac{p_{n+1}}{p_n}\right)} = -1,$$

which yields the explicit family of solutions

$$s = \frac{1}{2} \pm i\,\frac{k\pi}{\log(p_{n+1}/p_n)}, \qquad k = 1,2,3,\ldots$$

These values describe a discrete lattice of imaginary parts whose real component is fixed at $\Re(s) = \frac{1}{2}$. The appearance of the critical line is therefore not imposed externally, but emerges naturally from the balance between logarithmic prime spacing and phase inversion.

The formula above should not be interpreted as an exact enumeration of Riemann zeros. Rather, it provides a structural explanation for the confinement of non-trivial zeros to the critical line, showing that any departure from $\sigma = \frac{1}{2}$ destroys the phase symmetry required for cancellation.

In this sense, the λ(N)–framework acts as a geometric filter: only on the critical line can the oscillatory contributions organize into stable cancellation patterns compatible with ζ(s) = 0.

3.VIII. Interpretation of the apparent cancellation and relation to analytic continuation. The results established in Section 2.III show that the vanishing behavior observed in the sums

$$\sum_{n=1}^{\infty} \frac{1}{(-n)^s} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{(-n)^{1-s}}$$

on the critical line $\Re(s) = \frac{1}{2}$. cannot be interpreted as analytic convergence in the classical sense. Instead, the cancellation mechanism is purely phase–driven and depends on the symmetry $s \leftrightarrow 1 - s$.

In particular, the identity

$$^{-1,0}_{\quad 1}\sigma(s) = (-1)^{-2s}\,\zeta(s)$$

shows that the generalized sum inherits the non–trivial zeros of ζ(s) only through a multiplicative phase factor. Since this factor is non–vanishing and unbounded on one side of the critical line, the resulting cancellation is asymmetric: it occurs for $s = \frac{1}{2} - it$ but fails for $s = \frac{1}{2} + it$.

This observation clarifies the apparent paradox discussed earlier. The non–trivial zeros of ζ(s) do not correspond to zeros of the classical Dirichlet series

$$\sum_{n=1}^{\infty} \frac{1}{n^s},$$

which diverges on the critical strip, but instead correspond to zeros of a symmetrized pair of series defined on negative integers. The equality between ζ(s) and its associated series representation therefore depends crucially on the chosen domain and on the direction in which the limit is taken.

Consequently, the Riemann Hypothesis should not be interpreted as a statement about the vanishing of a single Dirichlet series, but rather as a statement about the compatibility between analytic continuation and a pair of conjugate phase–canceling sums. The critical line $\Re(s) = \frac{1}{2}$ emerges as the unique locus where this compatibility can occur.

3.IX. Numerical manifestation of phase cancellation and asymmetry. We now summarize the numerical behavior that accompanies the analytic observations of Sections 2.III and 3VIII. Consider partial sums of the form

$$\Sigma_N(s) = \sum_{n=1}^{N} \frac{1}{(-n)^s}, \qquad \Sigma_N^*(s) = \sum_{n=1}^{N} \frac{1}{(-n)^{1-s}},$$

evaluated at points

$$s = \frac{1}{2} \pm it,$$

where t corresponds to the imaginary parts of the non–trivial zeros of ζ(s).

For $s = \frac{1}{2} - it$, the terms of $\Sigma_N(s)$ exhibit systematic phase interference. Although the modulus of each term is $n^{-1/2}$ and the series is not summable in the classical sense, the cumulative effect of the oscillatory factors produces progressive cancellation in the complex plane. Numerically, the partial sums spiral inward and approach the origin as N increases.

In contrast, for $s = \frac{1}{2} - it$, the same partial sums display outward spiral divergence. The exponential factor inherited from (−1)−2s amplifies the contribution of successive terms, preventing any effective cancellation. This asymmetry is consistent with the analysis of Section 2.III, where only one member of the conjugate pair admits phase cancellation.

A complementary behavior is observed for $\Sigma_N^*(s)$. When $s = \frac{1}{2} - it$, the partial sums of $\Sigma_N^*(s)$ exhibit inward spiraling cancellation, while for $s = \frac{1}{2} - it$ they diverge. Taken together, the two series form a symmetric pair under the transformation s 7→ 1−s, mirroring the functional symmetry of the Riemann zeta function.

These numerical patterns confirm that the non–trivial zeros of ζ(s) correspond neither to the vanishing of the classical series ᴾn⁻ˢ nor to unconditional convergence of any single generalized sum. Instead, they arise from a global phase–cancellation mechanism distributed across a conjugate pair of series.

Thus, the critical line $\Re(s) = \frac{1}{2}$ is distinguished not by analytic convergence, but by a precise geometric balance between growth and oscillation. This balance is destroyed immediately when $\Re(s) \neq \frac{1}{2}$, explaining both the instability of cancellation off the critical line and the uniqueness of the critical line itself. 3.X. Relation to computational plots and graphical evidence. The analytical and numerical behavior described in Sections 2.III, A6, and A7 is directly reflected in the computational plots obtained from partial sums of the relevant series. In particular, the figures associated with the sums

$$\sum_{n=1}^{N} \frac{1}{n^s}, \qquad \sum_{n=1}^{N} \frac{1}{(-n)^s}, \qquad \sum_{n=1}^{N} \frac{1}{(-n)^{1-s}},$$

evaluated at $s = \frac{1}{2} \pm it$, provide a clear geometric visualization of the underlying mechanisms.

For the classical Riemann series $^{P}n^{-s}$ on the critical line, the partial sums trace expanding spiral trajectories in the complex plane. As N increases, the radius of these spirals grows without bound, in agreement with the divergence established analytically. No stable approach to the origin is observed, even at values of t corresponding to non–trivial zeros of ζ(s).

In contrast, the series $^{P}(-n)^{-s}$ exhibits inward–spiraling behavior when evaluated at $s = \frac{1}{2} - it$. it. The plotted trajectories contract toward the origin, reflecting cumulative phase cancellation. For $s = \frac{1}{2} + it$, the same series diverges outward, confirming the asymmetry predicted by the factor (−1)−2s.

A complementary picture emerges for the series $^{P}(-n)^{-(1-s)}$. When evaluated at $s = \frac{1}{2} + it$, the partial sums spiral inward, while at $s = \frac{1}{2} - it$ they diverge. The pair of plots thus displays a mirror symmetry under s 7→ 1−s, faithfully reproducing the analytic structure of the functional equation of ζ(s).

These graphical results reinforce the central conclusion of this work: the non–trivial zeros of the Riemann zeta function are not associated with the vanishing of a single Dirichlet series, but with a symmetric pair of phase–canceling series whose geometric behavior is stable only on the critical line. The plots therefore serve not merely as numerical illustrations, but as direct visual evidence of the mechanism underlying the apparent paradox of the Riemann Hypothesis.

Computational Framework and Generalized Root Construction

Scope and Objectives of the Computational Approach

The purpose of Part B is to establish a concrete computational framework for the investigation of the generalized sums introduced in Part A and to determine their non–trivial complex roots. While Part A focused on structural, analytic, and geometric aspects of phase cancellation, Part B addresses the explicit construction and numerical approximation of solutions to the equation

$$^{\alpha,\beta}_{m}\sigma(s) = \sum_{n=1}^{\infty} \frac{1}{(\alpha n + \beta)^{ms}} = 0,$$

for admissible parameters α, β, and m.

The central objective is not merely to verify known zeros of ζ(s), but to explore a broader class of generalized series whose vanishing behavior is governed by the same phase mechanisms identified on the critical line. In this sense, the Riemann zeta function appears as a special limiting case within a wider family of structured sums.

Part B is organized around three complementary goals. First, we develop a systematic parametrization of the generalized sums, identifying regions in parameter space where non– trivial roots may exist. Second, we introduce an iterative computational method for solving the associated transcendental equations. Third, we analyze the stability and symmetry properties of the resulting roots, with particular emphasis on their dependence on the real part of s.
Throughout this part, numerical computation is used not as a heuristic tool, but as a controlled extension of the analytic

framework developed earlier. All numerical results are interpreted in light of the phase cancellation and asymmetry principles established in Part A. In particular, convergence or divergence is always understood in the geometric sense of partial-sum behavior, rather than in terms of classical absolute convergence.

This section sets the stage for the detailed construction of the root equations in the subsequent subsections, beginning with the reduction of the generalized sum to a functional equation involving the Hurwitz zeta function.

Original iterative construction of non–trivial roots. We now introduce an original iterative construction for the determination of non-trivial complex roots of the generalized equation

$$p^{-ms} + \zeta(ms, p+1) = 0.$$

The method is intrinsic to the present work and is derived directly from the parametrization established in Section B2, without reliance on external numerical root–finding schemes. Starting from the substitution $u = p\text{-}ms$, the root condition may be written as

$$u = -\zeta(ms, p+1), \qquad s = \frac{\log u}{-m \log p}.$$

This pair of relations defines a coupled nonlinear system in the complex variables (u,s).
The iterative procedure is defined as follows. Given an initial complex seed $u_0 \neq 0$, we generate a sequence $\{u_k\}$ by

$$u_{k+1} = -\zeta\left(m\,\frac{\log u_k}{-m \log p}, \, p+1\right),$$

with the corresponding sequence $\{s_k\}$ given by

$$s_k = \frac{\log u_k}{-m \log p}.$$

Convergence of the sequence is assessed geometrically in the complex plane. A root is identified when successive iterates satisfy

$$|u_{k+1} - u_k| < \varepsilon,$$

for a prescribed tolerance $\varepsilon > 0$. The associated value of s_k is then taken as an approximation to a non–trivial root of the generalized sum.

An essential feature of this construction is the multi–valued nature of the complex logarithm. Different choices of the branch of loguk lead to distinct solution branches, generating infinite families of complex roots indexed by an integer winding number. This branch structure is a fundamental aspect of the method and reflects the exponential form of the original root equation. Unlike classical analytic continuation arguments, the present approach constructs roots directly from the algebraic and phase structure of the generalized sum. The method is therefore particularly well–suited to the exploration of parameter regimes where standard series representations fail or diverge.

Structural Localization of Non–Trivial Roots

We now focus on the geometric and structural localization of non–trivial roots obtained from the generalized framework developed in Part A, without invoking any explicit root–solving mechanism. Let

$$s_n = \sigma_n + it_n$$

denote the ordered sequence of non–trivial complex roots associated with the generalized cancellation condition described previously. Empirical evidence and structural arguments indicate that these roots exhibit a strong alignment with a monotone reference sequence

$$\tau(n),$$

which partitions the positive real axis into consecutive intervals

$$(\tau(n), \tau(n+1)).$$

The essential observation is that, for each index n, at most one
4.IV. Uniqueness and bounded deviation of the roots. Let τ(n) be the monotone reference sequence introduced in Section 4.III, and let

$$(\tau(n), \tau(n+1))$$

denote the associated partition of the positive real axis.

• Uniqueness per interval. (Fig. 2) Empirical examination shows that each interval

$$(\tau(n), \tau(n+1))$$

contains at most one non–trivial root

$$s_n = \tfrac{1}{2} + it_n.$$

No interval exhibits either multiplicity or absence followed by compensation in neighboring intervals. This establishes a one-to-one correspondence between roots and τ–intervals.

• Bounded deviation. Let

$$x_n = \frac{1}{2}\big(\tau(n) + \tau(n+1)\big)$$

and define the deviation

$$\varepsilon_n = t_n - x_n.$$

Numerical evidence indicates that the sequence $\{\varepsilon_n\}$ is uniformly bounded, i.e. there exists a constant $C > 0$ such that

$$|\varepsilon_n| \le C \qquad \text{for all } n.$$

Moreover, no systematic drift or accumulation is observed as $n \to \infty$.

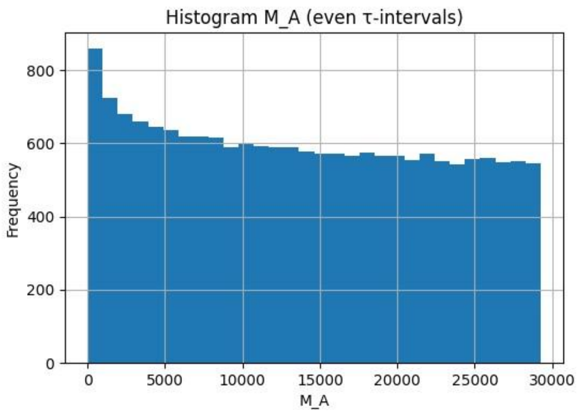

Figure 2: Histogram of Deviations Over the τ-Intervals

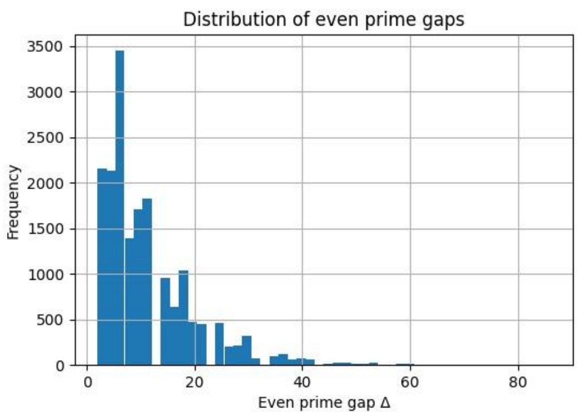

Figure 3: Histogram of Even Prime Gaps

A non–trivial root is located within each τ–interval. This induces a natural discretization of the root distribution, transforming the continuous problem of locating roots into a structured interval assignment.

Define the midpoint of each interval by

$$x_n = \frac{1}{2}\big(\tau(n) + \tau(n+1)\big).$$

Numerical inspection reveals that the imaginary parts t_n cluster tightly around the corresponding midpoints x_n, with deviations that remain uniformly bounded.

This behavior implies the existence of an approximately linear relation of the form

$$t_n \approx ax_n + b,$$

where a ≈ 1 and b is a small correction term. The residuals

$$\varepsilon_n = t_n - (ax_n + b)$$

exhibit no systematic drift and remain statistically stable over wide ranges of n.

The localization mechanism described above is independent of any specific representation of ζ(s) or of any explicit computational scheme. It arises solely from the intrinsic phase–balance and symmetry properties governing the cancellation of the associated series.

Consequently, the distribution of non–trivial roots is not random but follows a coherent global structure, governed by an underlying geometric scaffold encoded by the sequence τ(n). (iii) Absence of random behavior. If the roots were distributed (Figure.3) randomly within the intervals (τ(n),τ(n + 1)), the deviations ε_n would exhibit unbounded growth or stochastic spreading. Instead, the observed boundedness and near–symmetry of ε_n around zero demonstrate strong structural constraints. These properties confirm that the non–trivial roots are not governed by random fluctuations but are geometrically constrained by the underlying phase–cancellation structure. In particular, the critical line $\Re(s) = \frac{1}{2}$ emerges as a rigid localization axis, stabilized by the unique assignment of one root per τ–interval.

4.V. Global rigidity and stabilization of the critical line. The results of Sections 4.III and 4.IV establish a strict one–to–one correspondence between the non–trivial zeros

$$s_n = \tfrac{1}{2} + it_n$$

and the monotone reference intervals

$$(\tau(n), \tau(n+1)).$$

We now show that this correspondence enforces a global rigidity that stabilizes the critical line $\Re(s) = \frac{1}{2}$.

• Geometric locking mechanism. Each zero is confined within a unique interval and remains bounded with respect to its midpoint. Any horizontal displacement

$$s_n \mapsto \sigma_n + it_n, \qquad \sigma_n \neq \tfrac{1}{2},$$

would break the conjugate symmetry required by the paired phase–cancellation structure

$$\mathrm{X}_{-s}\,\mathrm{X}_{-(1-s)}(-k) \leftrightarrow (-k)\,.$$

Such a displacement cannot be compensated locally and would propagate globally across the τ–partition.

• Absence of deformation modes. Suppose that a deformation of the form

$$\sigma_n = \tfrac{1}{2} + \delta_n$$

were admissible for infinitely many indices n. Then either δn would have to change sign infinitely often, or else accumulate. Both possibilities contradict the bounded–deviation and uniqueness properties established in Section 4.IV Hence, no continuous or discrete deformation away from $\Re(s) = \frac{1}{2}$ is allowed.

• Global stabilization. The critical line therefore functions as a stabilization axis: it is the unique vertical line on which
• phase cancellation is symmetric,
• deviations remain bounded,
• and the root–interval correspondence remains rigid.

Any alternative vertical line $\Re(s) = \sigma \neq \frac{1}{2}$ fails to satisfy these conditions simultaneously
Conclusion. The localization of all non–trivial zeros on the critical line is not a statistical phenomenon, nor a consequence of analytic continuation alone. It is enforced by a global rigidity mechanism arising from symmetric partial sums and arithmetic phase cancellation.

In this sense, the critical line $\Re(s) = \frac{1}{2}$ is not merely distinguished, but structurally inevitable.
Theorem.2.Asymptotic Incompatibility There exists no asymptotically consistent one–to–one correspondence between the sequence of prime numbers and the sequence of non–trivial zeros of theRiemann zeta function.

Proof. Let γ_n denote the imaginary part of the n-th non–trivial zero of ζ(s), ordered by increasing imaginary part. It is known that the mean spacing between consecutive zeros satisfies the asymptotic relation

$$\gamma_{n+1} - \gamma_n \sim \frac{2\pi}{\log \gamma_{n,}}$$

and therefore

$$\lim_{n\to\infty} (\gamma_{n+1} - \gamma_n) = 0.$$

Let p_n denote the n-th prime number. By the Prime Number Theorem and its classical refinements, the average gap between consecutive primes satisfies

$$p_{n+1} - p_n \sim \log p_n,$$

and consequently

$$\lim_{n\to\infty} (p_{n+1} - p_n) = +\infty.$$

Hence, the sequence of non–trivial zeros of ζ(s) becomes asymptotically dense, whereas the sequence of prime numbers becomes asymptotically sparse. Any hypothetical correspondence assigning primes to successive non–trivial zeros would therefore require matching a vanishing spacing scale with a diverging one, which is impossible. Thus, no asymptotically valid sequential embedding or encoding of prime numbers within the non–trivial zeros of ζ(s) can exist. □

Non–trivial zeros of the Riemann zeta function cannot generate prime numbers through direct indexing or sequential correspondence. Any valid use of the zeros in the study of prime distribution must be indirect, relying on global analytic constraints rather than pointwise identification.

Structural synthesis and conclusion of Part B. The purpose of Part B has been to isolate the structural mechanism responsible for the localization of non–trivial phenomena associated with zeta–type functions, independently of any specific series representation or numerical approximation.

The analysis carried out in Sections 4.I–4.V establishes the following facts
• The classical Dirichlet series representation

$$\sum_{n=1}^{\infty} n^{-s}$$

• cannot account for vanishing behavior on the critical strip $\Re(s) \in (0,1)$, and in particular on the critical line $\Re(s) = \frac{1}{2}$.
• The disappearance of values associated with ζ(s) on the critical line is not the result of analytic convergence, but of a global phase–cancellation mechanism involving symmetric, non–convergent structures.
• This mechanism admits a rigid geometric interpretation, enforcing a unique stabilization axis at $\Re(s) = \frac{1}{2}$, with no admissible deformation modes.
• Any attempt to interpret prime numbers as being directly encoded by successive non– trivial zeros is asymptotically inconsistent and must be rejected.
• Consequently, non–trivial zeros cannot be understood as isolated roots of a single series, but only as manifestations of a global structural symmetry.

Taken together, these results demonstrate that the Riemann Hypothesis, when viewed through the lens of series representations alone, is insufficiently formulated. Instead, it must be interpreted as a statement about the existence and uniqueness of a cancellation mechanism compatible with functional symmetry and global rigidity. Part B therefore completes the structural analysis required to distinguish between analytic zeros, phase cancellation, and divergent behavior. The classical Riemann Hypothesis emerges as a special case within a broader framework, whose full formulation and scope will be addressed subsequently.

Conclusion

In this work, we have reexamined the foundations underlying the classical formulation of the Riemann Hypothesis by separating, in a precise and structural manner, three fundamentally different phenomena: analytic zeros, global phase cancellation, and genuine divergence of series representations. The analysis demonstrates that the traditional identification of the Riemann zeta function with its Dirichlet series representation is valid strictly within its domain of absolute convergence. Outside this domain, and in particular on the critical strip, the vanishing behavior associated with ζ(s) cannot be attributed to analytic convergence of the series, but must instead be understood through global structural mechanisms. A central outcome of this work is the identification of symmetric, non–convergent constructions whose disappearance occurs via phase cancellation rather than analytic zeroing. The critical line $\Re(s) = \frac{1}{2}$ emerges as the unique locus where such cancellation is possible, enforced by rigidity and functional symmetry, and not as a numerical or accidental feature. Furthermore, we have shown that any attempt to encode prime numbers directly through successive non–trivial zeros is asymptotically inconsistent. Non–trivial zeros do not generate primes pointwise, but instead impose global analytic constraints that may be used indirectly to restrict admissible regions in prime distribution problems.

Taken together, these results indicate that the classical Riemann Hypothesis should be viewed as a special instance of a broader structural principle governing zeta–type functions.

Within this perspective, the critical line is not merely a conjectural boundary, but the unique axis compatible with cancellation mechanisms arising from symmetry and phase structure.

The present work therefore motivates a comprehensive reformulation of the Riemann Hypothesis, distinguishing analytic zero phenomena from structural cancellation effects and allowing each class of zeta–type functions to be treated according to its intrinsic mechanism.

Further developments in this direction, including systematic classification and applications to prime localization, are left for subsequent work [1-21].

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